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SPACE TIME MOTION

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AN HISTORICAL INTRODUCTION
TO THE GENERAL THEORY
OF RELATIVITY

By

PROFESSOR A. V. VASILIEV

Translated from the Russian by

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With an Introduction by

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TO THE
PHILOSOPHER AND POET
MY SON
AND MY FRIEND
PROFESSOR N. A. VASILIEV

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"To expect to reorganise our ideas of Time, Space and Measurement without some discussion which must be ranked as philosophical is to neglect the teaching of history and the inherent probabilities of the subject." A. N. WHITEHEAD

TRANSLATORS' NOTE

THE reader will see that Professor Vasiliev quotes from many authors, Greek, Latin, English, French, German and Italian as well as Russian. So far as possible these quotations have been traced and retranslated from the original language and a reference to the source has been added—if there is an English translation a reference has generally been made to that. Some references to Russian books have been omitted.

Professor Vasiliev has been kind enough to give assistance on certain points and has suggested a few slight alterations and some of the additional notes. With his permission a few phrases have been omitted.

CONTENTS

	PAGE
DEDICATION	v
TRANSLATORS' NOTE	vii
INTRODUCTION	xi

CHAPTER I. FROM PYTHAGORAS TO NEWTON

1. <i>The doctrines of the Pythagoreans</i>	1
2. <i>Zeno and Protagoras</i>	5
3. <i>Democritus and Plato</i>	9
4. <i>The atomism of Democritus</i>	11
5. <i>The cosmology of Plato</i>	15
6. <i>The doctrines of Aristotle</i>	18
7. <i>The influence of the Aristotelian doctrines</i>	21
8. <i>Mediaeval ideas of space and time</i>	25
9. <i>Renaissance physics</i>	29
10. <i>Descartes and Newton</i>	33
11. <i>The rise of the new mechanics</i>	39
12. <i>Galileo and Newton</i>	46

CHAPTER II. THE CLASSICAL NEWTONIAN MECHANICS

1. <i>Newton and mathematical physics</i>	53
2. <i>The mathematical formulation of the principle of Relativity</i>	58
3. <i>Criticism of absolute space, time and motion. George Berkeley</i>	63
4. <i>The successes of mathematical physics</i>	69
5. <i>Space and time according to Kant</i>	72
6. <i>The views of Helmholtz</i>	81
7. <i>The empiriocriticism of Ernst Mach</i>	84

CHAPTER III. NON-EUCLIDEAN GEOMETRY

1. *Criticism of Euclidean geometry* . . . 92
2. *Non-Euclidean geometry* . . . 100
3. *Extension and motion* . . . 109
4. *The properties of geometrical space* . . 115
5. *Metrical representation of space* . . 120
6. *Geometrical and physiological space* . . 124

CHAPTER IV. THE SPECIAL THEORY OF RELATIVITY

1. *The electro-magnetic theory of light* . . 131
2. *The Lorentz hypothesis* . . . 133
3. *The mathematical expression of the principle of Relativity* . . . 141
4. *The relativity of space and time* . . . 145
5. *Mass as a form in which energy appears* . 156
6. *The world as a four-dimensional manifold* . 164
7. *The mathematical expression of the special theory of Relativity* . . . 169

CHAPTER V. THE GENERAL THEORY OF RELATIVITY

1. *Absolute motion and inertia* . . . 176
2. *The principle of equivalence* . . . 182
3. *The general theory of Relativity in the light of the latest discoveries in physics and astronomy* . 190

CHAPTER VI. THE PHILOSOPHIC MEANING OF THE THEORY OF RELATIVITY . 198

INDEX OF NAMES . . . 229

INTRODUCTION

IT is a great pleasure to me to have the opportunity of introducing to British readers the admirable book by Professor Vasiliev on "Space, Time, Motion." To pursue abstract studies in Russia during recent years has required a high order of devotion and endurance. I became acquainted with the author of the following pages in Petrograd in 1920, and was deeply impressed by his zeal in maintaining mathematical studies throughout adverse circumstances. Of this, the present volume is a proof.

Very appropriately, the author begins his historical account with Pythagoras, to whose emphasis on integers the moderns have been gradually returning. Such matters as continuity and irrationals, which his school found inexplicable, have been expressed by the pure mathematicians in terms of integers. The maxim '*natura non facit saltum*'—which, like many other groundless dogmas, was thought indubitable because it was in Latin—has been thrown over by the theory of quanta, which has given a new importance to integers in physics. The theory of relativity, however, with which Professor Vasiliev is mainly concerned, has not been affected by the reaction against continuity, though there is nothing in it which is in any way contradictory to that tendency. The philosophy underlying the theory of relativity is only in part modern, and this

book enables the reader to see its roots in the past.

The theory of relativity—like all sound science—is not *based* upon any philosophical doctrine, but merely upon the search for a comprehensive account of observed facts. Nevertheless it gives support to a philosophical doctrine, which is partly old, partly new. The old part of the doctrine is that space and time consist merely of relations; the new part is, that there are not two sorts of relations, one constituting space, the other constituting time, but only one, which will be differently analysed into a spatial and a temporal part according to the point of view of the observer. A good deal of confusion has been introduced into popular conceptions by the name ‘relativity,’ since Einstein’s innovation consists not in the relational theory but in the unification of space and time.

As every one knows, Newton believed in absolute space and absolute time, related to bodies as a hat-rack is related to the hats that are hung on it. Leibniz, on the contrary, considered space and time to be merely systems of relations between things and events, but he regarded force as something absolute, since otherwise the dynamics of his day became unworkable. This landed him in inconsistencies from which Newton’s doctrine was free. Berkeley—to whom we are

glad to find Professor Vasiliev doing justice—practically invented the modern psychology of space-perception in his “New Theory of Vision” —a work which ought to have awakened Kant from his ‘dogmatic slumbers’ more thoroughly than his cursory reading of Hume. It became obvious that there was a conflict between the theoretical requirements of Newtonian dynamics and what was empirically discoverable. Only relative motions and relative accelerations could be observed, yet dynamics demanded a distinction between relative and absolute acceleration. This inner contradiction was only resolved by the *general* theory of relativity.

The first to tackle the problem seriously was Mach. He considered that only observable phenomena should enter into the laws of an empirical science such as physics. Whether the earth rotates once a day from west to east, as Copernicus taught, or the heavens revolve once a day from east to west, as his predecessors believed, the observable phenomena will be exactly the same. Nevertheless, Newtonian dynamics will explain Foucault’s pendulum and the flattening of the earth at the poles if the earth rotates, but not if the heavens revolve. This shews a defect in Newtonian dynamics, since an empirical science ought not to contain a metaphysical assumption which can never be proved or disproved by observation

—and no observation can distinguish the rotation of the earth from the revolution of the heavens. This philosophical principle, that distinctions which make no difference to observable phenomena must play no part in physics, has inspired a good deal of the work on relativity. It is called phenomenalism, and is advocated by many modern writers.

Although this principle is one which might seem self-evident to any empiricist, it cannot be said that even the general theory of relativity applies it strictly. Indeed, it is probably impossible to construct any physics without infringing it. Einsteinian physics, no less than the physics of Newton, is realistic. It assumes that a physical process is something which many people can observe, and that only what all their observations have in common belongs to the process itself. As soon as I assume the existence of any other observer than myself I have abandoned the principle of adhering only to observable phenomena, since a world in which other people were phantoms would appear to me exactly like one in which they had an independent existence. Indeed the very word 'observable' covers a confusion. It is assumed, for example, that those processes which we do not examine under the microscope would, if we did examine them, be found similar to those which we have

so examined. The 'observable' is always interpreted as being wider than the observed; yet this interpretation, when scrutinised, is found to involve assumptions which cannot be either verified or refuted by any actual observation, and which, therefore, are contrary to the principle of phenomenalism.

Phenomenalism, therefore, must be regarded as an unattainable ideal, towards which we can gradually approach, but which we cannot hope to reach. There is a contrary principle, to the effect that the general body of physical science must be maintained. It may be that this is merely a pragmatic principle, without theoretical justification; on the other hand, the success of physics suggests that some theoretical justification must be possible. In practice, physics moves between these two opposite principles: so long as the science can be maintained intact, the elimination of anything not observable represents a philosophical advance. In this respect, Einstein is far ahead of his predecessors. But it would be claiming too much to maintain that there is *nothing* in his physics as unobservable as Newton's absolute space and time. Moreover the greatest merits of his system are quite unconnected with this philosophical question.

The scientific merit of Einstein's theory lies in the explanation, by a uniform principle, of

many facts which are unintelligible in the Newtonian system. The philosophical interest lies chiefly in the substitution of the single manifold, space-time, for the two manifolds, space and time. Let us consider these two aspects successively.

As every one knows, the special theory of relativity arose out of such facts as the negative result of the Michelson-Morley experiment. It appeared that the velocity of light is the same with respect to all bodies, no matter how they may be moving. If a train is going 60 miles an hour, and a motor-car on a parallel road is going 40 miles an hour in the same direction, the relative speed of the train from the point of view of a person in the motor-car is only 20 miles an hour. But if a ray of light is sent out in a given direction with a velocity of 300,000 kilometres per second, and a β -particle is sent out in the same direction with a velocity of 290,000 kilometres per second, the ray of light, from the point of view of an observer on the β -particle, is still moving with a velocity of 300,000 kilometres per second, not of 10,000 kilometres per second as might be expected. If a fly touches the surface of a stagnant pool, ripples move out from the point of contact in concentric circles, but the fly, if it moves on, does not remain at the centre of the ripples. If, however, the ripples

travelled with the velocity of light, the fly would seem to itself to remain at the centre, however fast it might be moving. And we could not say that it was mistaken. These facts are unintelligible with our ordinary views of space and time, but they are explained by the special theory of relativity, which also explains many other facts, such as the [apparent] increase in the mass of a body in very rapid motion.

The general theory of relativity is interesting rather for theoretical than for empirical reasons. Nevertheless there are the three well-known facts which it explains: the motion of the perihelion of Mercury, the amount of bending suffered by light in passing near the sun, and the shift of spectral lines towards the red in a strong gravitational field. (The last of these remained for some time uncertain, but appears to be now established.) All these are inexplicable on Newtonian principles. The first had long puzzled astronomers; the other two were predicted by Einstein and subsequently observed.

There are therefore ample empirical grounds for accepting both the special and the general theory. If this were not the case, it would not be worth while to undertake the overhauling of our fundamental notions which they involve. As it is, this overhauling is quite necessary. The extent of it may still be more or less doubtful;

Dr Whitehead, for example, does not believe it necessary to adopt a non-Euclidean geometry. But at the lowest estimate, very great and important changes are required in our ordinary notions.

It is not altogether easy to explain, in non-technical language, what is meant by the substitution of space-time for space and time, but a few illustrations may help to make the matter clear. Suppose you travel from London to Edinburgh in eight hours; commonsense would say that your departure from London and your arrival in Edinburgh were separated by two sorts of intervals, 400 miles in space and eight hours in time. It turns out, however, that this division into two kinds of intervals would be differently made by an observer moving rapidly relatively to the earth, and does not represent any physical fact. It is obvious that, if all motion is relative, the distance between two places at different times depends upon the body to which the motion is referred. Relatively to the railway carriage, you have not moved at all in travelling from London to Edinburgh. Relatively to the fixed stars, you have travelled, not merely 400 miles, but all the distance covered by the earth in its orbit during eight hours. There is no theoretical reason for preferring one of these standards of reference to another; therefore the distance you have travelled is a matter of convention.

It is more difficult to see that the time taken on the journey is also, within limits, a matter dependent upon the way the observer is moving. I say 'within limits' because, if the traveller himself makes the time of his journey eight hours, no other observer will make it less, though people moving rapidly in relation to the traveller will make it more. All this does not have to do with anything psychological; it has to do with clocks and measuring rods, and is wholly in the realm of physics. It was formerly thought that we knew what was meant by the simultaneity of two events in different places, but a little consideration shews that different observers will form different judgments as to whether two distant events are simultaneous. The obvious definition is to say that two distant events are simultaneous if an observer exactly half-way between them sees them simultaneously. But observers who are moving in different ways will form different estimates of what is half-way between the two events, and will therefore arrive at different results as to whether the events are simultaneous or not. Any two events which one observer, using the most correct methods, judges to be simultaneous, will be judged by a second observer to be one before the other, and by a third to be the other before the one. We cannot say that one of these is right and the others wrong;

all are only reporting, quite correctly, facts dependent upon their own motions. Neither distance in time nor distance in space are properties of two events considered in themselves, but both are dependent upon the way in which the observer of the events is moving.

There is, however, between two events which are not too far apart, a certain relation called 'interval,' which is independent of the observer and has physical significance. If, for some observer, the two events appear simultaneous, the interval is what this observer would judge to be their distance in space. In this case, the interval is called 'space-like.' If some observer could travel from one event to the other, what he would judge to be the time between the two events is their interval; in this case the interval is called 'time-like.' The only other possibility is that the events are parts of one light-ray, in which case their interval is zero. This notion of interval, as generalised in the general theory of relativity, is the only quantitative spatio-temporal relation between events which is independent of the motion of the observer. It is obvious that physical laws, if there are any, must be independent of the observer; hence all ought to be derived from the formula for the interval. In regard to gravitation, this is what Einstein's theory does.

As Professor Vasiliev very truly says, 'one of

the most important aspects of this theory is undoubtedly the fusion of geometry and physics.' It is a matter of taste whether we say that geometry is merged in physics or that physics is merged in geometry. The geometry, however, is not quite that of our school days. It is the geometry of space-time, not of space, and in Einstein's system it is non-Euclidean. Forces have disappeared, and are replaced by local peculiarities of the space-time. A place where there is matter is a place in the neighbourhood of which space-time is crumpled in a certain way. There is no longer any need to believe in matter in the old sense; there are only strings of events and peculiarities in the structure of space-time, which must itself be regarded as a plenum of events. The philosophy involved in this point of view has not yet been thoroughly worked out. It will have to replace both substance and cause by differential relations of neighbouring events, and what we have conceived as the motion of a piece of matter will be a string of connected events like the successive notes of a tune. The philosophical consequences, so far as they are definite, are given by Professor Vasiliev in his last chapter, but, as he recognises, many further consequences remain to be worked out.

It must not be supposed that, in the theory of relativity, space and time become 'subjective,' in

the sense, for example, of Kant's philosophy. The old realism said: Two bodies have a spatial distance, and two events have a distance in time. The old idealism said: These two relations, spatial and temporal distance, do not belong to the bodies or events in themselves, but depend upon the way in which they are perceived. The modern theory says: Both distance in space and distance in time must be taken to be between *events*, but neither is between two events alone; each is relative to some standard string of events which can be interpreted as the motion of a body of reference. This is still within the physical world, and does not involve a percipient. The only intrinsic quantitative relation between two events is their spatio-temporal interval. The arguments as to 'subjectivity' are not affected one way or the other, but they must now be applied to space-time, not to space and time separately. The new physics, like the old, proceeds on a realistic assumption, but the possibility of an idealistic interpretation remains exactly as it was: it is neither facilitated nor rendered more difficult by anything in the modern theory. The only change is that the controversy must be about space-time, not about space and time separately.

The general theory of relativity is probably the greatest synthetic achievement of the human intellect up to the present time. It sums up the

mathematical and physical labours of more than two thousand years. Pure geometry from Pythagoras to Riemann, the dynamics and astronomy of Galileo and Newton, the theory of electromagnetism as it resulted from the researches of Faraday, Maxwell, and their successors, are all absorbed, with the necessary modifications, in the theories of Einstein, Weyl, and Eddington. So comprehensive a synthesis might have represented a dead end, leading to no further progress for a long time. Fortunately, at this moment quantum theory has appeared, with a new set of facts outside the scope of relativity physics. This has saved us, in the nick of time, from the danger of supposing that we know everything.

BERTRAND RUSSELL.

CHAPTER I

FROM PYTHAGORAS TO NEWTON

L'histoire de l'esprit humain est la plus
grande réalité ouverte à l'esprit humain.

RENAN.

I. THE DOCTRINES OF THE PYTHAGOREANS

Plato in one of his dialogues contrasts the geometry created by the Greeks as the doctrine of the eternal with the practical science of the Egyptians.

Proclus in his review of the history of geometry says that PYTHAGORAS was the first to make of his occupations a real geometrical science, since he first regarded it from a higher point of view and studied its theorems 'not by the senses but by the way of cogitation.'

Is this true? Did mathematics as a theoretical science first arise not in the ancient river civilisations of the East, but in the flourishing Greek colonies on the Italian shore of the Mediterranean? This question is as yet unsettled, but it is usually answered by a recognition of that special gift for logical thought which Renan called 'le miracle grec.' There is another question too, which will probably never be solved. What share of the discoveries ascribed by the Greek writers to the legendary personality of the philosopher and sage, the political and religious reformer, belonged in actual fact to Pythagoras himself,

and what to his pupils and their successors? One thing, however, is not in doubt, and that is that in the history of human thought the school of Pythagoras holds an unique place. There is no doubt that in the opposition between the teaching of the school of Pythagoras and that of the Ionian philosophers we see the first source of the struggle between the two tendencies whose opposition is the *leit motif* in the history of philosophy. On the one hand, Thales, Anaximenes and Heraclitus, with their reliance on the senses and their inclination to take for reality what could be seen and felt—water, air and fire; on the other, the first idealist philosophy, the first attempt to find in the laws of number the reality underlying the changeful phenomena of the world. Let us listen, for example, to a Pythagorean of the fifth century, Philolaos, a contemporary of Democritus and Socrates, and the author of a book “On Nature”:

All things that are known have number, for without it nothing can be understood or known. . . .*

The nature and harmony of number does not suffer a falsehood, for it has no kinship with them.†

In the Pythagorean school an irrational number was first discovered and this was taken as the basis for the theoretic teaching of numbers. By

* Stobaeus, *Ecl.* I. (2 Ch. 3 M.), 456 (Wachsmuth, p. 188).

† *Ibid.* (18 Ch. 13 M.) 8 (Wachsmuth, p. 16).

the Pythagorean discovery of the numerical ratio between the length of the string and the pitch of the note, mathematical physics brought the harmony of Nature into relation with the harmony of number. In the Pythagorean school the earth was first removed from its central position, the universe was bounded by the sphere of the fixed stars, and the motions of the heavenly bodies were connected with the theory of regular solids. It was obvious that the school could not fail to raise questions as to space, time and motion, and we know that in the beginning of the fourth century the Pythagorean Archytas of Tarentum, a contemporary of Plato (who visited him on his travels) had already definite answers to the questions of the nature of space and time. From his answers which have reached us with the commentaries of Simplicius, we must conclude that he adhered to the teaching as to the reality of absolute place, which we shall meet throughout the whole history of the question.

Since what is moved is moved in a certain place and doing and suffering are motions, it is plain that place, in which what is done and suffered exists, is the first of things.

Since everything which is moved is moved in a certain place, it is plain that the place where the thing moving or being moved shall be, must exist first. Perhaps it is the first of all beings, since everything that exists is in a place and cannot exist without a place.

If place has existence in itself and is independent of bodies,

then as Archytas seems to mean, place determines the volume of bodies.*

His definition of time is no less interesting: 'Time is the number of a certain movement and in its widest sense the interval of the natural order of the universe.'†

The last words are connected with the doctrine of the periodicity of the universe. A similar doctrine apparently arose independently in India. 'The life of *Brahma* lasts a hundred years consisting of days and nights. The day of Brahma (Kalpa) is the day of Nature, in the night of Brahma Nature rests.'‡

There is abundant evidence that the idea of the periodicity of the Universe, the notion of the Great Year, was one of the favourite ideas of the Pythagoreans. 'The Pythagoreans believed,' says Aristotle's pupil, Eudemus, 'that I shall again speak with you, that I shall hold in my hands the same staff, and that you will sit in the same place and listen to me. . . . All things including time will be identical.'§

* *Commentary on the Categories*, c. 9 (Kalbfleisch, pp. 357, 361).

† *Ibid.* (Kalbfleisch, p. 350).

‡ Albiruni, quoted by Reinaud ("Mémoire Géographique Historique et Scientifique sur l'Inde. . . ." *Mémoires de l'Académie des Inscriptions et Belles Lettres*, t. XVIII. 1849, pp. 351, 352), from a ms. in the Bibliothèque Nationale, ancien fonds, No. 584, fol. 32.

§ *Commentary on Aristotle's Physics*, Book IV. 12 (Diel, p. 732).

An echo of this doctrine of the Pythagoreans appears to be the well-known perfect or 'bridal' number of the Eighth Book of Plato's "Republic," a number which lawgivers ought to know, for the number of the years of the period of happy and unhappy marriages is connected with it.*

This perfect number is in Duhem's view also the number of Archytas of Tarentum, inherent in the nature of the Universe, the period intervening between two world catastrophes, after each of which, as the Pythagoreans hold, the Universe returns to its former state.

2. ZENO AND PROTAGORAS

The importance of the problems agitating the minds of the Pythagoreans could not fail to call forth that intense intellectual activity which, in its turn, was bound from the nature of the ques-

* As to Plato's bridal number a plentiful literature has grown up. Loria (*Le scienze esatte nell' antica Grecia*, 1902) gives sixteen interpretations (from 50 to 1,296,000) which were ascribed to it. Some think it equalled 10^4 , others see in it the holy number of Babylon 60^4 . The latter (and, according to that competent critic Duhem, better) interpretation is that of Dupuis.

The Platonic number after which all beings, whether divine or mortal, return to their former state is the multiple of the cycle of the short Metonic cycle (19 years) and 10,000 years of a period in which, according to Plato, each soul is renewed again and is therefore equal to 760,000 years. [Author's note.]

A period is four years. There is a full discussion of Plato's number in Adam, *The Republic of Plato*, Appendices to Book 8.

tions concerned to lead not only to differences but to diametrically opposite opinions. Just as in the nineteenth century, the school of Hegel split into a right and a left wing, and the extreme spiritualism of the one side evoked the materialism of Feuerbach, Marx and Engels; so evidently the Pythagorean point of view pushed to extremes (things are copies of numbers or even things are numbers) produced the opposite doctrine within the school itself. To the philosophy of the 'Many,' the Eleatics opposed the philosophy of the 'One.' The founder of this school, PARMENIDES, in his poem expounds both *truth*, which is recognised by the reason alone, and *opinion*, which is given only by the senses. *Truth* is the idea of the world as a Single Being, continuous and immovable, a universe which, though specific and final, nevertheless fills the whole of space. *Opinion* is learning about the earth, water, air, the heavenly bodies, their movements and origin. The connection of this opinion with the Physics of the Ionians, and particularly the Pythagoreans, has now been established. The 'Truth' of Parmenides, his doctrine of the One, could not fail to call forth opposition from the right wing of the Pythagoreans, who held, like Philolaos, that everything known must be connected with number.

Plato's dialogue, "Parmenides," has preserved for us the memory of the talk which took place

about 400 B.C. in the Athens of Pericles between the old man Parmenides, his favourite pupil, Zeno, and the youth Socrates. Zeno had just published a book in defence of the ideas of his teacher. The opponents of the theory of the One continuous and immovable Being deduced from it absurd and ridiculous conclusions. Zeno, 'the father of dialectic,' proves that the doctrine affirming the opposite leads to conclusions no less absurd and self-contradictory. In the 'aporiai,' or dilemmas, which have come down to us, thanks to Aristotle, Zeno shows that the doctrine of the Many leads to the assertion that each thing is at one and the same time infinitely great and infinitely small. In the history of the questions of space, time and motion, Zeno's aporiai as to movement and space hold an important place. One of these, called the 'Stadium,' sets out to prove that the measurement of time was impossible, as any interval of time may be equal to any part of it. However obscure the statement of this aporia may be, it is evident that the argument led to the necessity of taking into account the relativity of motion.

The aporiai of Zeno furnish incontestable proof that the philosophic thought of the Greeks was already in the fifth century directed to those problems of the relation between the discrete and the continuous, the finite and the infinite, which

again arose in the seventeenth and eighteenth centuries in connection with the discovery of the infinitesimal calculus, and which latterly have gained a new meaning in the philosophy of mathematics through the genius of GEORG CANTOR. Paul Tannery was right when he said that Zeno, though not a mathematician, had by the subtlety of his dialectic made possible the strictness and accuracy of the results and ideas which distinguish Greek mathematics. This constitutes the great service of Greek dialectics. It is, however, natural, on the other hand, that the dialectical struggle between the theories of the One and the Many which divided the two wings of the Pythagorean school should lead to scepticism in relation to metaphysical theories. This scepticism found its most brilliant exponent in PROTAGORAS, who opposed to fruitless abstract reasoning the necessity of turning to living actuality, to immediate sensations. Goethe, on the one hand, Nietzsche, on the other, saw in Protagoras the thinker who had most in common with their own views, 'Man is the measure of all existing things in so far as they exist.'* This saying of Protagoras has been given different interpretations. All agree, however, that to the philosophy of the Pythagorean school which strove to find the hidden reality behind the world

* Diogenes Laertius, IX. 51.

of phenomena, Protagoras first opposed the philosophy of sensualism and phenomenalism.

3. DEMOCRITUS AND PLATO

A writer of the third century B.C., a follower of the scepticism of Protagoras, *Sextus Empiricus*, tells us that the doctrine of Protagoras was refuted both by Democritus and Plato. In the history of philosophy hitherto it has been customary to oppose the atomic doctrine of Democritus to Plato's theory of ideas, and to see in Democritus the father of materialism, and in Plato the protagonist of the idealist philosophy. The legend was even believed that Plato, indignant at the materialism and atheism of Democritus, or from envy of his fame, burnt all his works. A careful study of the sources and the spirit of both doctrines, however, leads many authors to the opposite view that there is a connection between the teachings of Democritus and Plato. The fact that the philosophies of Democritus and Plato have many features in common is explained by the strong influence of the Pythagoreans. Both put the search for the 'Logos,' the rational basis of phenomena, in opposition to the naïve materialism of the Ionians and the sensualism of Protagoras. While for the Pythagoreans, numbers were the rational basis hidden behind the world of phenomena, so for

Plato the eternal and immutable was not a single whole and continuous substance, as Parmenides thought, but is distributed among a world or system of separate entities existing outside space and time, the Platonic 'ideas.' In the last stage of the Platonic metaphysics, under the influence of the Pythagoreans, the ideas become idea-numbers. On the other hand, the atoms of Democritus appear, in the illuminating phrase of one historian of philosophy, as the 'materialisation' of numbers. Both Plato and Democritus accordingly gave a high place to mathematics. The Platonic dialogue "Philebus" first gave expression to the thought that each science is only a science in so far as it contains mathematics, an idea afterwards repeated in the same or different words by Leonardo da Vinci, Galileo, Descartes and Kant. The aim of geometrical astronomy was, according to Plato, to 'save phenomena,' to find law and regularity in the complicated movements of the stars. The "Ephodikon" of Archimedes, recently found in Constantinople, shews that it was Democritus who first discovered the theorems relating to the volume of the cone and the pyramid. If we bring this fact into connection with the one phrase which has reached us from the mathematical works of Democritus, we are bound to conclude that he was the inventor of the method of indivisibles. This method,

which was used after him by Eudoxos of Cnidos and Archimedes, and was rediscovered only in the seventeenth century by Italian geometers and formulated by Cavalieri in his "*Geometria indivisibilibus continuorum nova quadam ratione promota*," which appeared in 1635, that is almost twenty-one centuries after Democritus.

When we find that views, at first sight diametrically opposed, are so closely connected, it is not surprising that the space theories of Plato and Democritus also are reminiscent of the space theory of *Archytas of Tarentum*. Space for Democritus is independent of bodies as in the case of Archytas. Space, however, for Archytas was at the same time active, and this doctrine was reflected in Plato's view of space as the 'foster-mother of all becoming.'*

4. THE ATOMISM OF DEMOCRITUS

In spite of the similarity of their doctrines and starting points, Democritus and Plato went different ways in the development of their metaphysical theories. This difference was chiefly due to their different attitude towards the study of nature. While Plato, entirely engrossed in questions of metaphysics, ethics and politics, attached little importance to experiment or observation, Democritus, to quote the words of Pliny in his

* *Timaeus*, 52 D.

“Natural History,” ‘passed his whole life among experiments.’ As a result of this mode of life, Democritus composed an enormous number of works—thirty or forty—on different subjects, mathematics, physics, natural history, ethics, aesthetics, grammar, strategy and farming. Unfortunately, not a single work of all this encyclopaedia of Greek learning has survived. All that has reached us are phrases torn from their context, chiefly dealing with questions of ethics. Thanks mainly to Aristotle, we are able to reconstruct Democritus’ brilliant anticipation—the atomic theory. The first attempt at the mechanical explanation of the phenomena of nature could only be the creation of a deeply philosophic spirit and the penetration of the thinker enthusiast who ‘had rather discover the true theory of knowledge than become ruler of Persia.’*

In a somewhat altered form the theory of Democritus† was popularised by Epicurus and afterwards, in the form given to it by Epicurus, was expounded in the poem of Cicero’s contemporary, *Titus Lucretius Carus*, “De Rerum Natura.”

* Eusebius, *Preparatio Evangelica*, Book xiv. 27, quoting Dionysius of Alexandria.

† The question of the relation between the philosophies of Democritus and Epicurus was the subject of the doctor’s dissertation of Karl Marx. [Author’s note.]

‘The real basis of things is atoms and the void ; all else is opinion, illusion.’* ‘Only in opinion is there cold or warm ; in reality there is only atoms and the void.’* It was thus that Democritus formulated the fundamental doctrines of his metaphysic. Lucretius expounds the theory of space and time held by the atomistic school in the following lines:

All nature then, as it exists by itself, is founded on two things: there are bodies and there is void in which these bodies are placed and through which they move about. For that body exists by itself the general feeling of mankind declares, and unless at the very first belief in this be firmly grounded there will be nothing to which we can appeal on hidden things in order to prove anything by reasoning of mind. Then again, if room and space which we call void did not exist, bodies could not be placed anywhere nor move about at all to any side.†

Time also exists not by itself, but simply from the things which happen. The sense apprehends what has been done in time past, as well as what is present and what is to follow after. And we must admit that no one feels time by itself abstracted from the motion and calm rest of things.‡

The history of the atomistic hypothesis is one of the most interesting and important pages in the history of human thought. In the middle of the fifth century A.D. one of the Christian fathers,

* Sextus Empiricus, *Adversus Mathematicos*, Book VII. 135-139. See also Diogenes Laertius, *Vita Pyrrhonis*, Book IX. 72.

† Book I. ll. 419-427.

‡ Book I. ll. 459-463.

Jerome, exclaimed triumphantly: 'Who now reads Aristotle except idle old men? Who knows the books, nay even the very name of Plato?' A yet deeper oblivion overtook the works of Democritus, Epicurus and Lucretius, which were sharply opposed to the views of Christian theology. In the Middle Ages it was only in one Arab School of Philosophy (Mutakallimun) that the atomistic hypothesis was taken as the basis of philosophy and pushed to its ultimate conclusions. This school held not only that matter was discrete, but that space, time and motion were also discrete. In Western Europe only little by little under the influence of Arab and Hebrew thinkers, from the experiments of alchemists and the observations of doctors, the atomistic doctrines of the nature of matter again began to grow up and met with sharp opposition and persecution on the part of the ruling scholastic philosophy. In 1348 the works of Nicolas Autricurius, in which the phenomena of nature were referred to the union and disjunction of atoms in space, were burnt in Paris. There, too, in 1624 it was forbidden on pain of death to attack Aristotle, and some bold spirits (Bitault, Villon and de Claves) were banished because they had undertaken publicly to defend certain theses, one of which asserted that everything consists of indivisible atoms. These persecutions, however,

could not prevent the spread of atomism. Cardinal Nicolas Cusanus (1401-1464) as early as the fifteenth century calls to mind the fundamental ideas of atomism. In 1646, in Milan, the physician Magnenus printed a book with the title "*Democritus reviviscens, sive de Atomis.*"

In 1649 Gassendi expounded the doctrines of Epicurus in Paris. Henceforward, the importance of the atomic theory, both in exact science and the history of thought in general, was no longer in doubt. The beginning of the twentieth century, which was distinguished in the history of physics by the development of the theory of electrons and the discovery of the Quantum theory, still further increased its significance.

We must now return to Plato and Democritus, and especially the influence of Democritus on the philosophic thought of Greece. The respect in which Aristotle held Democritus may be judged from the fact that in all his works he gave the most space to polemics directed against him.

5. THE COSMOLOGY OF PLATO

The relation of PLATO to Democritus has always been a puzzle to historians of philosophy. Plato never mentions his name in any one of his dialogues, but recently an author (Jenssen), who had made an exhaustive study of the "*Timaeus*," pointed out that a considerable part of this dia-

logue was certainly written under the influence of the teaching of Democritus. The "Timaeus," the only one among Plato's many dialogues devoted to questions of cosmology and physics, expounds Plato's views through the mouth of the Pythagorean Timaeus, the ruler of the colony of Locri—a further indication of the connection of Plato's views with the doctrines of the Pythagoreans. The "Timaeus" is one of Plato's most obscure dialogues, but it has had an enormous and prolonged influence on the physics and cosmology of Western Europe. Down to the middle of the twelfth century, as Duhem says, in the schools of Western Europe, logic and dialectic were studied in the "Organon" of Aristotle, but Aristotle's physics were unknown. All views on cosmology and physics were derived almost exclusively from the "Timaeus," and the commentaries on it, chiefly the work of Neo-Platonic philosophers, Plotinus and others. In the "Timaeus," the Jewish and Christian Fathers (Philo of Alexandria and Augustine) found some resemblance to the cosmology of the Book of Genesis, and seriously discussed the question whether Plato could have been acquainted with the Bible.

The theory of space, expounded by Plato in the "Timaeus," is in close connection with the fundamental doctrines of his philosophy. The true reality which Democritus saw in the atoms is

found, according to Plato, in the ideas. Only the ideas, immutable, indestructible, not to be perceived by the senses, and only to be known by the reason, constitute Being. On the other hand, things which become and perish, which are subject to change, partake in a very slight degree of reality. It is they which are perceived by the senses. Between these two categories there is a third intermediate one, perceived only by mathematical cogitation. This third category is space, 'the foster-mother of all things existing.'* Plato thinks that outside the universe which he, like the Eleatics, takes to be bounded and spherical, there is an infinite *empty* space in which the universe is placed. But, in contradistinction to Democritus, he does not allow a void within the universe. Space, he says, in one of the most puzzling passages of this obscure dialogue, is filled with geometrical figures, triangles of the two types which form the faces of the four regular solids (the cube, eikosihedron, octahedron, and tetrahedron). The four elements, earth, water, air and fire, correspond to these four polyhedra. The fifth regular solid (the pentagonal dodecahedron) was used by the creator of the universe, the *Demiourgos* as the basis of his general plan. Evidently there is much in common

* *Timaeus*, 88 D.

between the triangles of Plato and the atoms of Democritus.

6. THE DOCTRINES OF ARISTOTLE

Such, then, were the two theories of Plato and Democritus which, in the fifth century B.C., answered the problems of space and its relations to the general problems of physics. In the fourth century a new answer to these same problems was given by a pupil of Plato—ARISTOTLE—who, from 355 to 342 B.C., expounded to the pupils in his "Lyceum" the whole encyclopaedia of philosophy and natural science which was within the reach of the Hellene of the fourth century. Unlike Democritus and Plato, Aristotle broke with the tradition of the Pythagoreans—the identification of Being with the properties of numbers. He blames his master for identifying philosophy with mathematics, which were only important as an instrument. Aristotle gives enormous importance to the processes of induction and classification; and his philosophy may be correctly called a philosophy of a biological type. His doctrine of space, time and motion is explained mainly in his "Physics," but also in his "De Coelo," "De Motu Animalium" and "Problemata Mechanica." (It should be added that Aristotle's authorship of the last two has been disputed.) The great historical importance of

Aristotle's doctrines makes a more detailed treatment of them in place here.

'Aristotle's "Physics" is one of the most striking systems constructed by the human mind. All questions which the ancients might raise as to the heavens and their movements, as to the elements and their changes, it answers more precisely and more fully than ever had been done before, and it combined all these logically into a theory in comparison with which the earlier doctrines were mere rough sketches.' In these words, Duhem ("Le Système du Monde," vol. I, p. 242) sums up his excellent analysis of Aristotle's "Physics."

The "Physics" is divided into eight books. The first book (on the "Origins of Being") and the second ("On Nature") explain Aristotle's characteristic doctrines as to matter and form, the four causes, chance and necessity. With the third book begins physics proper or the doctrine of being in so far as it is capable of motion. Aristotle gives a more general sense to the term 'motion' than that which it has borne since the time of Descartes. Motion in our sense of the word means movement from place to place. For Aristotle, coming into being and annihilation and any quantitative or qualitative change (diminution or increase; getting cold or hot) are different aspects of the general concept motion. Natural

science deals with the problem of motion, quantity and time; but since motion, quantity and time can all be either finite or infinite, Aristotle devotes almost the whole of the third book to the question of infinity. It is the fourth book which is specially interesting on the doctrines of space and time. It deals with problems of space, time and the void. Aristotle gives his definition of place and time and expounds in detail his objections to the 'void' of the Atomists. The three following books (the fifth, sixth and seventh) contain the theory of motion. Like quantity and time, motion is continuous. Taking this as his starting point, Aristotle disproves Zeno's aporiai. At the end of the seventh book Aristotle puts the problem as of the relation between a motive force and motion. He deduces from experiment a proposition which was taken as the basis of dynamics for many centuries until first Galileo and his school, and then Newton, disproved it and replaced it by another proposition which has been taken as the basis of classical mechanics. Finally, the eighth and last book is intimately connected with Aristotle's philosophy. The problem of the eternity of motion is connected with that of the source of motion ('primum mobile')—the eternal being which moves all and is not moved.

This book of the *Physics*, like the "*Timaeus*,"

has great historical importance. Plato's "Demiourgos" and Aristotle's "Primum mobile" fused together with the Jehovah of the Bible, created the idea of the Deity which it was the task of Christian theology to develop. The "Physics" of Aristotle has a wide range and varied contents. We can only deal with the history of a few of the questions which it raised—the problem of 'place' and time and the relation between force and motion. We shall attempt to show how Aristotle's definition of place in its successive stages of development gave rise to the definition of 'absolute space,' which is one of the fundamental postulates of Newton's "*Naturalis Philosophiæ Principia Mathematica*"; and how the relation between force and motion laid down by Aristotle, which was based on a mistaken experiment, was transformed in Newton's immortal book into the 'second law of motion.'

7. THE INFLUENCE OF THE ARISTOTELIAN DOCTRINES

Aristotle's doctrine of space is related to his doctrine of place. One of Zeno's *aporiai* denied the existence of space; in Plato's "*Timæus*" the idea of space becomes part of the concept of matter. Democritus, on the other hand, found the existence of the void a necessary condition for motion. For Aristotle the displacement of a

body does not prove the existence of the void, but only the existence of place.

Where now there is water, when it has come out as if from a vessel, there will be air. Since then different bodies fill the same place, this seems to be something different from all the things that are in it or come into it. Where air is, water was before, so that it is plain that place is something and the space out of which and into which the bodies moved is something different from both bodies.*

But the movement of bodies not only proves the existence of place but also demonstrates its properties.

For each thing if it is not hindered is carried to its own place, some upwards some downwards. There are parts and kinds of place, upwards and downwards and the remainder of the six directions. Right and left, upwards and downwards, do not only exist in relation to ourselves. For this happens in accordance with the position in which we are turned, so that the same thing is often up and down and right and left and behind and before. But each exists separately in nature. For upwards is not by chance, but the way fire and light things are carried, and likewise downwards is not by chance, but the direction of heavy and earthy things.†

This is Aristotle's doctrine of 'natural places of bodies,' which is connected with his other doctrine of natural and forced motions. Each substance has its corresponding proper motion. Heavy bodies move downwards to the centre of the Universe; light bodies turn upwards. These

* *Physics*, Book iv. ch. i. 3, 4.

† *Ibid.* ch. i. 4.

rectilinear motions are the natural motions of the bodies of the sublunar world. The natural movement of heavenly bodies is in a circle, for that is the only motion which can continue for an indefinite time.

In proving the existence of space and demonstrating its properties, Aristotle is seeking for its definition. He tries to bring it into relation with one or other of his logical categories and comes first of all to the conclusion that the only possible definition of place is the following: 'The place of a body cannot be anything else than . . . the part in immediate contact with the body.'* If a solid body is put in water, the place of this solid body is the water which it immediately touches. That this definition is unsatisfactory is evident from the fact that it would follow from it that a ship at anchor in running water is in motion, for the water round it is changing.

Accordingly, therefore, place is not the medium in immediate contact with the body; for this medium can be in movement, while place must be motionless. In the search for the real place we must reach something motionless which surrounds on all sides both the body itself and all the moving bodies with which it is surrounded. So in the case of the ship, a motionless body of this kind both in respect of the ship and the

* *Physics*, Book IV. ch. IV. 12.

water it is in can only be the banks and the bottom of the river. Hence, we get the new definition of place to which Aristotle adheres: 'The first motionless part of the environment which we come across near a body, will be the place of that body,'* as well as the place of all other bodies which are included in this environment. It is easily seen that Aristotle's attempts to give a more accurate definition of place are trying to solve the fundamental problem of mechanics, which later took the form of the difference between apparent rest and motion, between absolute and relative motion.

In his book "De Motu Animalium" Aristotle says: 'Everywhere that a body is in motion there must be something which is motionless.'† But what is this motionless something in virtue of which alone we are able to judge whether other bodies are in motion or at rest? This is the question which for many centuries divided the philosophers who in general shared Aristotle's views and who were called the Peripatetics. This name ('walkers') they bore in common with Aristotle's own immediate pupils who had walked with him in the gardens of the Lyceum. One group of Aristotle's followers and commentators (the group to which belong Damascius and his pupil Simplicius of the sixth century, who were

* *Physics*, Book IV. ch. IV. 12. † Didot, vol. III. p. 518.

the nearest to Aristotle's time among the Greek commentators) held that it was not necessary that there should be any definite concrete motionless body.* 'Place seems to be the measure of the position of bodies, just as time was the number of the motion of things moving.'†

8. MEDIAEVAL IDEAS OF SPACE AND TIME

In the eighth and ninth centuries a new people—the Arabs—entered the intellectual field. Mesopotamia, the country between the Tigris and the Euphrates, became again, as it had been twenty centuries earlier, the centre of a culture whose influence stretched from Khiva and Balk on the one hand, to Cordova and Toledo on the other.

In Bagdad and Basra, in Toledo, people were hard at work collecting books, translating them into Arabic and in writing commentaries. Duhem notes an interesting difference between the Greek and the Arab commentators on Aristotle. While the Greek commentators are satisfied with any hypothesis which 'saves phenomena,' however strange it may seem from the physical point of view, the Arab commentators judge a hypothesis from the point of view of its significance in actual

* There were several editions of *The Commentaries of Simplicius on the eight books of the Physics* in the sixteenth century. [Author's note.]

† Simplicius, *Commentary on Aristotle's Physics*, Book iv. Cor. de loco (Kalbfleisch, p. 627).

fact. For instance, the Greek and the Arab commentators take up a different attitude towards Ptolemy's theory of Epicycles and deferents. Almost all the Arab astronomers are against it.

One of the most important Arab philosophers whose influence on European thought lasted till the middle of the seventeenth century, was the twelfth century philosopher, Abulvalid Mahommed Ibn Rushd, or AVERROES (1126-1198), called the Great Commentator ('Averroe, che il gran Commento feo,' Dante says of him, and places him in the first circle of the *Inferno* together with Euclid and Ptolemy*). In his commentaries on Aristotle's "Physics" Averroes deals in detail with Aristotle's definitions of place and defends the position that there must be a concrete motionless body. This, as we see, is the exact opposite to the view held by Damasius and Simplicius, that the motionless boundary to which all motions are related is not a sensible palpable body, but an idea whose existence is demonstrated by ratiocination.

The recognition of an absolute motionless body, which even the Almighty could not move, was evidently a contradiction to the fundamental doctrine of Christian theology, an Almighty God. Accordingly the doctrine as to the concrete motionless centre of the world, together with

* *Inferno*, IV. 144.

many other of the doctrines of Averroes' philosophy, was recognised as heretical* and condemned in a meeting of the doctors of the Sorbonne, which took place in 1277, under the presidency of Etienne Tempier, the Bishop of Paris. The doctrine of Damascius and Simplicius was recognised as the only orthodox one. It is upheld and demonstrated in the books of Thomas Aquinas, whose teaching still forms part of the obligatory doctrines of the Catholic Church. Many of the other great schoolmen, *e.g.* William of Occam, Duns Scotus, Albert of Saxony, and the Paris School in general, maintained that the movement of bodies cannot be understood if we do not postulate a motionless term to which the position of bodies can be referred at any given moment. But, according to this doctrine, it was not necessary that the motionless term of comparison should have an actual concrete existence. The same school of Paris, however, continued to produce thinkers who shared the views of Averroes. They held that every movement needs a motionless term of comparison which existed not as a pure idea, but in a concrete form as one of the bodies of the Universe. Opinions, however,

* Later, Averroes, as the protagonist of infidels, was held to be the author of the book *On the Three Impostors*, which was also ascribed to Frederick II, Cardan, Boccaccio, Hobbes, and many others. [Author's note.]

were divided as to what should play the part of the motionless 'term.' Some (*e.g.* the schoolman, Jean de Jandun) shared the views of Averroes, and assigned this part to the earth. Other schoolmen (Michael Scot), taking as their basis certain passages in the Bible, maintained that behind the movable heavenly spheres of the astronomers, behind the ninth sphere, there was the 'Empyrean,' a tenth motionless sphere, and they held that here was the solution of this vexed question. This doctrine was formulated in the thirteenth century by Campanus of Novara, the famous translator of Euclid into Latin, in his book "The Theory of the Planets." The Empyrean, Campanus holds, is the place of all things, for it contains them all and is not itself contained in them. This doctrine is historically very important. In the "De Revolutionibus orbium cœlestium" of Copernicus which asserts that the earth moves round the sun, the assertion is also made that the first and highest of the heavenly spheres (that of the fixed stars) is motionless and is the place in the universe to which the movement and position of the stars must be related. The Paris school of terminalists and the English and German schoolmen connected with it still, in general, held the view of the Greek commentators. Meanwhile in the Italian universities, especially the University of Padua (the Latin quarter of the

rich and powerful Venetian Republic), the Peripatetics were the 'apes not only of Aristotle but also of Averroes' (an expression of Jean de Jandun, which he applied also to himself). The most famous opponent of Galileo, Cremonini, a professor at Padua (who died in 1631), was also a follower of Averroes. In Italy the Peripatetic doctrines, especially in the form given them by the Arabs, and in particular by Averroes, lasted longer than elsewhere; but it was through Italy that new tendencies appeared in the thought of Western Europe and that other doctrines of Greek philosophy were re-born.

9. RENAISSANCE PHYSICS

A modern historian of the theory of knowledge speaks ecstatically of the 'incomparable charm' of the Italian Renaissance.* But the re-birth of the culture of the ancients had at first chiefly an artistic and literary character. Some historians connect the beginning of the scientific and philosophic Renaissance with the name of NICOLAS CUSANUS (1401-1464); others with the Congress of Florence in 1439 which brought to Italy the philosophers and theologians of the Byzantine Empire. Under the influence of the Greeks, especially Gemistos Pletho, interest in Plato revived. The Florentine Academy took for its

* E. Cassirer, *Das Erkenntnisproblem*, vol. 1. p. 73.

object the diffusion of Plato's ideas, and celebrated the 7th November each year as the day of the birth and death of Plato. Nicolas Cusanus, like Plato, again declared mathematics to be the basis of all knowledge ('Knowledge is measurement'), and held mathematical infinity to be the only way of approach to the mystery of infinity. He also, as we have seen, saved the ideas of the atomistic philosophy from the oblivion into which they had fallen. Not only the teachings of Plato and Democritus but other doctrines of Greek philosophy made their way. People began to hear of the books of a philosopher of the sixth century A.D. whose importance in the history of mechanics, Duhem and Wohlwill have recently brought to light. This was John, nicknamed 'Grammar' or 'Philopon' (labour-loving), an Alexandrian Christian who belonged, however, by his philosophic convictions to the school of the Stoics. Ten centuries before Galileo he subjected many of Aristotle's wrong conclusions (as we shall see in the following paragraph) to a wholesome criticism. This, however, passed unnoticed, and Aristotle's mistaken statements exercised a hypnotic influence even on geniuses like Leonardo da Vinci, Copernicus and Kepler. We will only pause on his views concerning space.

In his "Commentaries" on Aristotle's "Physics" which were published in the Latin version

several times in the sixteenth century, Philopon criticises Aristotle's definition of place and gives his own.

'Place is not the contiguous part of the surrounding body. Place is an extension measured in three directions distinct from the bodies which are found in it and incorporeal in its nature.'*

Starting from this definition of place, Philopon denies the existence in place or space of a force directing bodies to their natural places. Bodies are directed to their places not in consequence of the properties of space but in consequence of their own 'appetites.'

Eleven centuries later, Philopon's views of space, accepted by the Italian natural philosophers, take a mystic colouring at the hands of the English Platonist, Henry More, and in this form influence Newton. Philopon's incorporeal space distinct from bodies appears under the title of 'absolute' space as one of the fundamental principles of Newton's "*Philosophiae Naturalis Principia Mathematica*." To return, however, to the Renaissance,—the rush of ideas new to Western Europe taken from Greek philosophy of the distant past was opposed by deep-rooted traditions. It was only by slow degrees that natural science in Italy freed itself and broke

* *Commentaries on Aristotle's Physics*, Book IV. ch. 4. Cor. de loco (Vitelli, p. 567).

with the doctrines of Aristotle and the Schoolmen. Thus the gifted *Cardan* (1501-1576), doctor, mathematician, and mystic, a man of immense learning, in his "De Subtilitate," adheres to the view that place is the part surrounding a body dividing it off from other bodies. Another humanist, Scaliger, however, in a book criticising Cardan, already, like the ancient Atomists, identifies space with the 'void.' The place of the material volume is identical with the geometrical volume of the three-dimensional space occupied by these bodies. For Scaliger, space and time are abstractions from matter or motion. For Telesius, who adhered to the philosophical views of the Eleatics, space and time have an independent existence which we perceive by the senses, but which we could apprehend and define. In particular Telesius, in his "De rerum natura," regards space as a primary existence remaining changeless and the same through all changes of the position or movement of separate things. Telesius emphatically opposed Aristotle's theory of natural places, which assigns different properties to different parts of space; he holds that space is the same at all points and in all directions. The "Pancosmia" of *Francesco Patricio* (1529-1597) is specially interesting in the history of the problem of space. Here we meet the contrast between physical and mathematical space,

and there is a special chapter dealing with the relation between the two kinds of space.

‘What is space which existed before the universe and which contains it? Is it only the faculty (aptitudo) for containing bodies or has it some material existence? Is it a substance or accident, is it corporeal or incorporeal? All these definitions do not apply to space because they are invented for things, and space appears in the last resort as an incorporeal, not a corporeal body (corpus incorporeum et non corpus corporeum).’*

The last eminent representatives of Italian physics are two thinkers who suffered for their convictions—*Campanella* (1568–1639) and *Giordano Bruno* (1548–1600). Their views on space are also very interesting. Campanella, reviving the doctrine of Patricio, makes space an independent spiritual being endowed not only with consciousness and the instinct of self-preservation, but also with immortality. Bruno’s doctrine also is reminiscent of Patricio. Bruno’s infinite space is filled with ether. This ether, the agent of the world soul, is not a body, for it cannot be felt, nor yet spirit, for it has extension.

10. DESCARTES AND NEWTON

The year Giordano Bruno died saw the beginning of the seventeenth century. The first

* Franciscus Patritius, “Nova de Universis Philosophia” (Venetis, 1593), *Pancosmia*, p. 65. Slightly condensed.

half of this century was the epoch of Gassendi (1592-1655), and Descartes (1596-1650), Kepler (1571-1630), and Galileo (1564-1642). Of these, the first two had an immediate influence on the development of the philosophic doctrine of space and physics generally. We shall therefore pause to consider them.

GASSENDI was an ardent supporter of the atomistic theories of Democritus, Epicurus, and Lucretius, and his view of space as the void was essentially the same as that of the ancient atomists. 'Illimitable spaces,' says Gassendi, 'existed before God created the world and will remain the same if He should destroy the world. For his own pleasure God chose a definite region in which to found the world, leaving the rest of space void. . . . If God translated the world to another place, space would not follow the world.'*

On the other hand, for DESCARTES there is no empty space, that is, space not filled with matter. Everything that has extension in space is body.

'Space or the place occupied by a body and the bodily substance contained in it, do not differ in matter but only in the manner in which we are accustomed to conceive them.'†

This identification of matter with space called

* Opera (1727), *Physicae*, Sect. I. Liber II. c. 1, vol. 1. p. 163.

† *Principia Philosophiae*, Part II. x.

forth confutations from Leibniz after Descartes' death and in his life-time formed the subject of a lively dispute between him and Henry More. HENRY MORE (1614-1687) and Cudworth (1617-1688) transplanted to English soil the doctrine of Italian natural philosophy and were distinguished representatives of that English Platonism which had a great influence on philosophic and religious thought in England. The mechanical explanation of all the phenomena of organic and inorganic nature was the root idea of Descartes' physics. This they held to be the road to atheism, and accordingly More, in his interesting correspondence with Descartes, opposes to Descartes' identification of matter and space the spiritualisation of space. He held that this idea, which, as we saw, originated in Italian natural philosophy, would prove a mighty weapon against materialism. In More's chief metaphysical work "*Enchyridion metaphysicum sive de rebus incorporeis*," printed in 1674, the doctrine of the spiritual nature of space changes into its deification. 'This infinite extent will seem to be something divine. It cannot be nothing as it has so many splendid attributes, such as the following, which are ascribed by Metaphysicians to the Supreme Being. One, Simple, Motionless, Eternal, Complete, Independent, Pre-existent, Subsisting of itself, Incorruptible, Necessary, Immense, Uncreate, Not bounded,

Incomprehensible, Omnipresent, Incorporeal, All pervading and embracing, Essential Being.’*

More’s metaphysic seems strange to us, but we see its enormous historical importance, if we compare it with the metaphysic of the creator of classical mechanics. NEWTON’S metaphysical views are most clearly formulated at the end of his “Optics.” ‘Does it not appear from phaenomena that there is a Being incorporeal, living, intelligent, omnipresent, who, in infinite space, as it were in his sensory, sees the things themselves intimately, and thoroughly perceives them and comprehends them wholly by their immediate presence to himself: of which things the images only, carried through the organs of sense into our little sensoriums, are there seen and beheld by that which in us perceives and thinks.’†

In the scholium to the third book of the “Principia,” which came out thirteen years after More’s book, Newton sketches the relation between God, Space, and Time, in the following words: [God] ‘is not eternity and infinity, but eternal and infinite; He is not duration and space, but He continues and is present. He continues for ever and is present everywhere and by con-

* *Enchyridion metaphysicum sive de rebus incorporeis*, Part I. cap. 8, sec. 8—abbreviated.

† Newton, *Optics*, Book III. Query xxix.

tinuing to be always and everywhere, He constitutes duration and space.'

It is difficult to deny that Newton's statements approach very closely to More's spiritualised space. Its relation to the genesis of Newton's ideas and consequently to the classical mechanics, and the connection of the idea of 'action at a distance' with More's teaching as to the motion of matter, is not less interesting.

More holds that the motion of matter can only be explained on the supposition that there is a spiritual substance which produces the motion, and this he calls the hylarchic principle or *Spiritus Naturae*. In the same way, action at a distance can only be explained by supposing that it takes place through incorporeal space filled with a spiritual substance, the organ or sensorium of God. This metaphysical atmosphere of the English Platonism in which Newton's metaphysical ideas arose and developed, influenced his definitions of absolute space, absolute time, and absolute motion. These definitions are given in the "Principia" in the scholium which precedes the axioms or laws of motion.

'I. Absolute true, mathematical time of itself and from its nature without relation to anything external flows equally; and its other name is duration.'

'II. Absolute space by its nature without relation to anything external always remains similar and

motionless. Relative space is a measure of this space or a certain movable dimension of it, which is defined by our senses by its position in regard to bodies, and is usually taken for motionless space.

‘III. Place is the part of space which a body occupies and is according to the space either absolute or relative.

‘IV. Absolute motion is the translation of a body from an absolute place to another absolute place; and relative motion is the translation from one relative place to another.’

Newton's definition marks an epoch in the history of mathematical physics and natural philosophy. In these last sections we have attempted to give a concise historical sketch of the change from Aristotle's physics and their confused definitions of place through the disputations of the Mediaeval Schoolmen and the doctrines of the Italian and English natural philosophers till we reach these definitions of Newton. In the following chapter we shall pause to consider the significance of these definitions and their connection with the philosophy of the present time.

For two centuries Newton's definitions were regarded by almost all thinkers as the unshakeable and changeless fundamentals of physics. In the later chapters, which deal with the special and the general theory of relativity, we shall meet with a theory which breaks away from these definitions.

II. THE RISE OF THE NEW MECHANICS

Let us return now to that important proposition of Aristotle which for twenty centuries formed the basis of the dynamics of the Peripatetic School.

This fundamental proposition was the result of a mistaken experiment. That this was a venial mistake, may, however, be seen from the fact that it passed unobserved, even by the creators of Statics, men who shewed great penetration in studying the principles and developing the science.

Leonardo da Vinci, Simon Stevinus, Roberval, Varignon, in the fifteenth, sixteenth and seventeenth centuries, all with one accord took the axiom of Aristotle as the foundation of their researches in Statics. Even in the beginning of the eighteenth century, Saccheri (Lobachevski's great predecessor and a subtle logician) in his "Neo-Statics" (1703), takes as his foundation the axiom of Aristotle. Even Galileo, the founder of modern dynamics, for a long time took this axiom as the starting-point for his works, and never formulated his disagreement with it. Only Newton definitely broke with it and replaced it by another which is still taken as the basis of mechanics.

The axiom is formulated by Aristotle in two

books whose authenticity is not disputed—"The Physics" (Book VII, chapter v) and "De Coelo" (Book III).

'Let A be the mover, B the object moved, C the distance, D the time taken. In the same time the same force A will move half B twice the distance C ; or the same C (distance) in half D (the time).'

In "Problemata Mechanica" (a work of doubtful authenticity, but if not by Aristotle it is apparently by one of his most immediate pupils) this axiom was applied to the deduction of the rule as to the equilibrium of the lever which Archimedes deduced from premises which had no connection with motion.*

Aristotle's axiom may be expressed algebraically by the formula

$$G = mv$$

where G is the force, m the mass or weight, and v the velocity. In other words, the velocity is Aristotle's measure of force. Accordingly, if the force ceases to act, that is, is reduced to zero, the speed also becomes zero, that is, the body comes to a state of rest. Thus the law of inertia was for Aristotle the law of the conservation of rest. But experience constantly shows that a

* Duhem upholds this opinion in his book *Les Origines de la Statique*, but it is disputed by Vailati, an Italian *savant* who died young. [Author's note.]

body continues to move even when the action of the force has ceased; this contradicts the axiom. The difficulty is obviated by asserting that the air which is put in motion by the moving body preserves this motion for a longer or shorter time, and gradually communicates it to the moving body. Aristotle's assertion is disproved by Galileo in his famous "Dialogue about the Two Systems of the World" (Second Day).

The history of pre-Galilean dynamics has not yet been worked out from its sources, many of which are hidden in the libraries of Europe in the form of rare books and MSS.

Only very lately many interesting facts have been discovered by Wohlwill and Duhem.* Wohlwill was the first to show the importance in history of dynamics of John Philopon, whom we mentioned in the last section. Philopon first definitely opposed the theory of the influence of the medium. Why should the hand which moves the stone have to touch it if the air does everything? Philopon also, by pointing out self-evident facts, attacks Aristotle's assertion that heavy bodies fall faster. The soundness of Philopon's views and his definition of space entitle him to be considered one of the most prominent thinkers

* Mach's *Mechanics*, though extremely important philosophically, is not up to date on the historical side. [Author's note.]

of the Middle Ages and a precursor of Galileo. Duhem shews the importance in the history of mechanics of the thirteenth century scientist, Jordanus Nemorarius, and especially a certain book written by an unknown pupil of his school whom Duhem calls the 'fore-runner of Leonardo da Vinci.' A brilliant record in the history of mechanics was left by Leonardo whom one historian of philosophy (Ravaisson) calls the 'founder of modern thought.' This epithet, however, might be applied to Nicolas Cusanus, of whom we spoke in the preceding section. We will pause for a moment to consider him before we deal with the great artist. Among the numerous works of Nicolas Cusanus, there is one called "De ludo globi" (probably a kind of billiards). Here he calls attention to the fact that a well-turned ball keeps its motion on a polished level surface for a long time. This he ascribes, not like the Mediaeval Schoolmen who followed Aristotle to the influence of the medium, but to the '*vis impressa*,' communicated force which the original blow gives to the ball. In this book we first find the phrase '*vis impressa*,' which afterwards was used in the sixteenth and seventeenth centuries and was introduced by Newton into his statement of his laws of motion.

The recognition of the place of Nicolas Cusanus in the history of philosophy, mathematics, and

mechanics, is comparatively recent. The importance of LEONARDO DA VINCI (1452-1519) in this sphere has not even yet been fully recognised. The creator of the "Last Supper," and "La Gioconda" was undoubtedly one of the great geniuses of all time. During his life he published nothing, and it was only in 1797 that Venturi called attention to the scientific importance of his MSS. These and his notebooks which have not yet been published in full, prove that he was interested in many different branches of knowledge. Whether in geometry, mechanics, hydraulics, astronomy, physics, natural history, anatomy, music, or strategy, he left the imprint of his genius on all. He studied mechanics with particular enthusiasm. 'Mechanics is the paradise of the mathematical sciences, because only in mechanics is it possible to taste the delightfulness of their fruits,'* is how Leonardo expresses his feeling for mechanics. So long as all Leonardo's MSS. are not published,† it is impossible to get

* "Fragmenti Letterari e Filosofici." *Pensieri nella Scienza*, LII. *Della Mechanica* (Barbera, 1899), p. 86. In J. P. Richter's edition it is No. 1155, vol. II. p. 289.

† A photographic facsimile of all the MSS. belonging to France has been published with a French translation in a fine edition. In 1893, thanks to the generosity of a Russian Maecenas, Sabashnikov, a sumptuous edition of the codex *Sul volo degli uccelli* (on the flight of birds) was published. The precious Codex Atlanticus at Milan, and other MSS. belonging to Italy are as yet unpublished. [Author's note.]

a complete view of all his achievements in the realm of mechanics. But we already know that his reflections on the equilibrium of the lever of the motion of bodies on an inclined plane led him to the theory of statical moments, and that he was on the way to the discovery of the famous principle of possible displacements which was laid down by Lagrange as the basis of his analytical mechanics. Was the history of mechanics influenced by the fact that Leonardo did not publish his works in his life-time and that his precious MSS. were partly destroyed and partly stolen? Duhem deals with this question in his "Origins of Statics" and comes to the conclusion that Leonardo's MSS. were in the hands of Cardan, and that Cardan* (whose name is associated with more than one plagiarism) made an extensive use of them. His book "De Subtilitate," which appeared in 1551, was translated into French five years afterwards and went through many editions in both languages in the sixteenth and early seventeenth centuries, Cardan probably borrowed most of his facts and reflections from Leonardo's MSS.

Thanks to this fact, the inspirations of Leonardo's genius were able to influence the further development of science both in Italy (Benedetti, Galileo,

* The solution of cubic equations which is called Cardan's was printed by him in the *Ars Magna* in spite of the promise which he had given to Tartaglia. [Author's note.]

and Torricelli) and in the Low Countries (Simon Stevinus).

Leonardo and Cardan and the Italian scientists of the sixteenth century who wrote on mechanics, all took the axiom of Aristotle as the starting point of their studies of motion. The law of inertia was for them the law of the conservation of rest. As to motion, some of them thought (taking as their basis the commentaries on Aristotle) that circular motion may continue for ever, while motion in a straight line must certainly come to an end. Telesius, for example, knows that the fall of a body is an accelerated motion, but explains this by the analogy of the tired traveller who hastens his journey as he approaches his resting-place.

The works of Benedetti have had, as Vailati has shown, an immense importance. First of all, he asserts, in contradiction to the Peripatetics, that similar bodies fall with the same speed independently of their dimensions and mass. Secondly, Benedetti knows that a stone tied to a string and turned quickly in a circle tries to get farther out, and that this explains the tension of the string. Finally, Benedetti distinguishes movement under the impulse of a momentary shock from movement due to influence of a continuous force, and uses this influence (*impressio*) to explain the increase of velocity of motion; that

is to say, for him force appears as a source not of velocity but of acceleration.

In these and many other passages in his works, Benedetti appears as the immediate precursor of Galileo, who knew him. His works lead us to the very threshold of classical mechanics. We are on the verge of the discovery of the laws governing the fall of heavy bodies, the law of inertia, the measurement of the force of acceleration, and the formulation of the principle of relativity.

12. GALILEO AND NEWTON

‘In order to discover the satellites of Jupiter, the phases of Venus and the spots on the sun, telescopes and perseverance were needed; but it needed an extraordinary genius (*génie extraordinaire*) to discover the laws of nature amid the confusion of events which take place before the eyes of us all, but the explanation of which had always eluded the investigations of thinkers.’ In these words Lagrange sums up the services of Galileo in the sphere of mechanics. His mathematical discoveries are expounded in two of his chief works. One of these had a tragic result for him—his condemnation by the Inquisition and his recantation. The “Dialogue about the two most important systems of the World Ptolemaic and Copernican” was intended to

confute the metaphysical and physical arguments which were brought against the system of Copernicus. When he is controverting the physical arguments which had been brought against the rotation of the Earth on its axis, when he was proving that experiments intended to shew the absence of this movement had no importance, Galileo draws a vivid picture of what goes on under the deck of an evenly-moving vessel.

The ship may move with any velocity you please but, if only its motion is uniform and it does not rock from side to side, you will notice no change in any of the things we have mentioned nor be able to judge from any of them whether the ship is moving or at rest. If your friend is standing on the prow and you on the poop, greater force need not be used in throwing something for the other to catch, than if your positions were reversed.

Fishes in a bowl of water do not swim with more difficulty towards the front than the backward part of the bowl, but come with equal ease to look for their food, put on any part of the edge of the bowl. Lastly the butterflies and flies flutter about making no difference (as to direction) and it does not happen that when they are tired with staying in the air they are more likely to settle on the wall looking towards the poop as if following the course of the vessel more swiftly. In the same way we shall see the smoke from a grain of lighted incense rising like a cloud and not moving more to one side than the other.*

* *Dialogo dove... si discorre sopra i due massimi sistemi de Mondo Tolemaico e Copernico*. Second Day (pp. 411, 412 of the Milan 1811 edition).

Though somewhat confused and indistinct, this is the principle of relativity of the classical mechanics which was afterwards formulated by Newton* in the following passage: 'The motions of bodies enclosed in a given space are the same relatively to each other whether that space is at rest or moving uniformly in a straight line without circular motion.'

In the same Dialogue, Galileo also asserts the law of the conservation of motion in a straight line: A ball proceeding from a gun, 'would follow its own motion in a straight line, which is the continuation of the direction of the gun if its own weight had not made it depart from that direction towards the earth.' In many other places in the same Dialogue and in his other work, "Talks and Mathematical Proofs in the two new departments of knowledge (mechanics and the laws of falling bodies)," Galileo uses the law of inertia, applying it to particular instances, but he never formulates it as a general principle. Such a formulation, however, would not have been a matter of great difficulty, and his pupil, Baliana, extracted from Galileo's exposition the principle that a velocity once attained is indestructible.

The conversations, "Discorsi," which appeared in 1638, give, as their full title shews, the famous

* *Principia*, Corollary v. to the Laws of Motion.

law of the fall of a solid body which is based on the new principle of acceleration. This principle, which is the exact opposite of Aristotle's axiom, is that acceleration (change of velocity) and not velocity itself is the result of the action of the *vis impressa*.

Thus the foundation of dynamics was laid and strengthened by ingenious experiments. Newton recognises the services of his famous predecessor; not so Descartes. Independently of Galileo, Descartes in the years 1617-19, studied the acceleration of a falling body, taking as his starting point the investigations of Cardan and Benedetti, who in their turn were influenced by Leonardo's inspiration. As we see from his letter to Mersenne, Descartes in 1629, before the publication of Galileo's book, was already acquainted with the law of inertia, and the law of uniformly accelerated motion under the influence of a constant force.

In 1644, Descartes published his "Principia Philosophiae." In this book, he deduces the laws of motion from a metaphysical doctrine, from the unchangeableness of the common primal cause of all motions which is none other than God Himself. He formulates the first law as follows:

The first law is that every given thing, in so far as it is simple and individual, remains of its own accord always in the same state and is not moved except by causes outside itself. Thus if any part of matter is square, we easily see that

it will always remain square, unless something comes from elsewhere to change its shape. If it is at rest we do not believe that it will ever begin to move, unless impelled by some other cause. Moreover, there is no more reason for us to think that once moved it will ever of its own accord and if it is not hindered by anything else, cease that motion.*

Descartes' second law is as follows:

The second law of nature is that each particle of matter considered by itself never has a tendency to move in oblique but only in straight lines. Many, however, are deflected by meeting other particles, and as we have said before in any motion there is in a certain way a circular movement† in all matters moving together. The cause of this rule is the same as that of the one before, that is to say, the immutability and the simplicity of the action by which God conserves motion in matter.‡

We have deliberately quoted Descartes' formula in its entirety. The reader may then judge for himself both its strength and its weaknesses. Its chief fault lies in the *a priori* judgments from which these laws are deduced. It must, however, be added that Descartes also gives references to experiments in support of the laws. Their strength lies in the fact that they are stated in a general form which Galileo did not give them. This is why Descartes adopts a patronising

* *Principia Philosophiae*, Part II. xxxvii.

† Here we have the origin of Descartes' theory of vortices, which was the logical consequence of the fact that Descartes, like Plato, identified space with matter. [Author's note.]

‡ *Ibid.* Part II. xxxix.

attitude to the works of Galileo. 'He' (Galileo), Descartes writes in 1659, 'considered the causes only of particular cases (*effets particuliers*) not following out the primal causes of nature (*les premières causes de la nature*) and therefore he built without a foundation.' Descartes' mind, which was prone to generalisations, did not value at their true worth those mathematical ideas which led Galileo, by integrating the formula for velocity, to obtain the law of motion; that is, to the connection between space and time. Both Descartes and Galileo set a high value on mathematics. The former saw in all sciences only a disguised chain of numbers, the latter said that the hieroglyphics in which the laws of nature are written are mathematical symbols and figures. Of the two, however, Galileo chiefly turned his attention to the laws of motion in continuous space and continuous time. He, rather than Descartes, was interested in questions of continuity, the infinitely great and the infinitely little.*

More than forty years after the death of Galileo and the publication of Descartes' "*Principia Philosophiae*," Newton in his "*Principia*" formulated the axioms or laws of motion and used them as the foundation of mechanics.

* In his *Discorsi* we meet with some paradoxes which were afterwards developed by Bolzano and which were used by Cantor as the starting point for his theory of quantity. [Author's note.]

Law I. Every body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled to change that state by impressed forces.

Law II. Change of motion (momentum) is proportional to the motive force impressed and takes place in the direction of the straight line in which that force is impressed.

Law III. Action and reaction are always equal and opposite; or the mutual actions of two bodies are always equal to each other and in opposite directions.

CHAPTER II

THE CLASSICAL NEWTONIAN MECHANICS

Sibi gratulenter mortales
Tale tantumque exstittisse
Humani generis decus.

*(Inscription on Newton's monument
in Westminster Abbey.)*

I. NEWTON AND MATHEMATICAL PHYSICS

NEWTON is not merely the founder of classical mechanics, that is, the science of those changes which Aristotle called local movements. He shares with Leibniz the glory of discovering the "royal road" for the study of motion in the Aristotelian sense, that is, of all continuous changes in the phenomenal world, whether we call them physical or psychical. This road is the differential calculus, which Newton called the method of fluxions. Newton undoubtedly used this method for deducing the consequences of the fundamental postulates of his "Principia," but he preferred to express his results in the geometrical form in which Archimedes and Apollonius had set forth their discoveries in geometry and statics. Only John Bernouilli and Euler had already clearly applied the differential calculus to the solution of mechanical problems. MacLaurin in his "Complete system of fluxions"

(Edinburgh, 1742) was the first to use the method of coordinates.

In his "Theoria motus corporum solidorum" (1765) Euler followed his example. Analytical mechanics, which reduces problems of motion and rest to the solution of problems of numbers and their functions, attained its highest development in Lagrange's classical work "Mécanique analytique" (1788); a book which still retains its importance.

Taking Cartesian coordinates and using the differential calculus, we obtain three differential equations for the free motion of a material particle:

$$\left. \begin{aligned} X - m \frac{d^2 x}{dt^2} &= 0 \\ Y - m \frac{d^2 y}{dt^2} &= 0 \\ Z - m \frac{d^2 z}{dt^2} &= 0 \end{aligned} \right\} \dots\dots(I)$$

where x, y, z are the coordinates of the particle relative to any fixed coordinate system K , m the mass of the particle, t the time, $\frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2}$ the components of the acceleration and X, Y, Z the components of the force. These three equations are the analytical expression of Newton's second

law. In the special case where no force acts on the particle, *i.e.* $X = 0$, $Y = 0$, $Z = 0$, it, as we see from the equations, moves uniformly in a straight line; that is, we have the Galileo-Newton law of inertia.

Let us now assume that as well as the fixed system K we consider another moving one K' , the directions of whose axes are parallel to those of the system K . At first let the coordinates of the moving system K' (whose coordinates are x_0, y_0, z_0 in relation to the system K) move according to some law, *i.e.* that the coordinates x_0, y_0, z_0 are functions of the time:

$$x_0 = f_1(t), \quad y_0 = f_2(t), \quad z_0 = f_3(t).$$

We learn from analytical geometry that between the coordinates x', y', z' , referred to the moving system K' , and the coordinates x, y, z we have the equations

$$x' = x - x_0, \quad y' = y - y_0, \quad z' = z - z_0$$

and thus:

$$\left. \begin{aligned} \frac{d^2 x'}{dt^2} &= \frac{d^2 x}{dt^2} - \frac{d^2 x_0}{dt^2} \\ \frac{d^2 y'}{dt^2} &= \frac{d^2 y}{dt^2} - \frac{d^2 y_0}{dt^2} \\ \frac{d^2 z'}{dt^2} &= \frac{d^2 z}{dt^2} - \frac{d^2 z_0}{dt^2} \end{aligned} \right\} \dots\dots(II)$$

Substituting in this expression the forces instead of the x , y , z coordinates, we thus obtain

$$\left. \begin{aligned} X' - m \frac{d^2 x'}{dt^2} &= m \frac{d^2 x_0}{dt^2} \\ Y' - m \frac{d^2 y'}{dt^2} &= m \frac{d^2 y_0}{dt^2} \\ Z' - m \frac{d^2 z'}{dt^2} &= m \frac{d^2 z_0}{dt^2} \end{aligned} \right\} \dots (III)$$

If in these equations we put $X' = Y' = Z' = 0$, that is, the case where the material particle is not acted on by a force, we see from the equations that the particle will not move uniformly in a straight line, that is, the law of inertia will not apply.

Let us suppose that an observer A is fixed to the origin of the coordinates of the moving system K' , he can determine by mechanical measurements the acceleration of some material particle $\left(\frac{d^2 x'}{dt^2}, \frac{d^2 y'}{dt^2}, \frac{d^2 z'}{dt^2} \right)$ in relation to the axes of the system K , and the forces X' , Y' , Z' acting on the material particle.

Then the equations (III) enable him to determine $\frac{d^2 x_0}{dt^2}, \frac{d^2 y_0}{dt^2}, \frac{d^2 z_0}{dt^2}$, i.e. his own proper motion. But this is not possible in one case, namely if the

system K' moves uniformly in a straight line in relation to the system K . Then we have

$$x_0 = ut + a, \quad y_0 = vt + b, \quad z_0 = wt + c \quad (\text{IV})$$

where u, v, w are the components of the constant velocity, and the second differential coefficients of x_0, y_0, z_0 in relation to t vanish.

In that case the equations of motion (I) and (III) coincide; and therefore the laws of motion of a material particle and also of a system of material particles are identical. It is as if the system were at rest. For the observer in the moving system K' all mechanical phenomena appear as Galileo in his dialogue vividly portrays.

Consequently we can express Galileo's principle of relativity mathematically as follows: the equations of motion do not change their form if for the variables x, y, z we substitute new variables x', y', z' related to them as follows:

$$x' = x - ut - a,$$

$$y' = y - vt - b,$$

$$z' = z - wt - c.$$

This does not bring out the assumption that the observer A in system K' uses the same time as the observer A in the fixed system K . In other words, the idea of an absolute time the same for both observers, introduced by Newton, is not made obvious. For what follows it is important to denote the time of the observer A by t and

that of the observer A' by t' , but to reckon that both times are equal and measured from one and the same moment, so that $t = t'$ and for the initial moment ($t = t' = 0$) we have $a = b = c = 0$.

Then we can state Galileo's principle in a new form as follows:

The equations of motion of a particle do not change if for the variables x, y, z, t we substitute new variables x', y', z', t' related to them by the equations:

$$\left. \begin{aligned} x' &= x - ut \\ y' &= y - vt \\ z' &= z - wt \\ t' &= t \end{aligned} \right\} (N)^*$$

or in other words, the variables x, y, z, t are connected with the variables x', y', z', t' by the transformations (N).

2. THE MATHEMATICAL FORMULATION OF THE PRINCIPLE OF RELATIVITY

We will now introduce the idea of a group; the explanation of the significance of this idea constitutes one of the most important results of contemporary mathematics.†

* N is used to remind the reader that this is the Newtonian transformation, later the author uses L for the Lorentz transformation and E for the Einstein transformation.

† In particular since Galois' investigations all the successes of higher algebra have been indissolubly connected with the theory of group transformations. Many important results

Let there be a series of operations S, S', S'' , Let SS' denote the operation consisting of the result of the two operations S and S' . (The operations may be changes of letters, *i.e.* objects generally, movements, reflexions, algebraic transformations, etc.) If any operation SS' (where S' may be identical with S) is found to be among our operations S, S', S'' , then the aggregate of these operations constitutes a *group*.*

If the number of separate operations which enter into the group is finite, then the group is called finite. The theory of permutations offers an example of such a finite group. The aggregate of the six permutations of three letters constitutes a group; for evidently the permutation derived from two successive permutations gives one of the permutations. Another interesting example of finite groups is offered by those movements which make a regular polyhedron to coincide with itself. In the case of an octahedron, for example, such movements will be rotations round

recently obtained in the theory of differential equations have also been due to the introduction of the idea of a group. Mathematical crystallography is also based on it. In chapter III the reader will learn the importance for geometry of the theory of groups. [Author's note.]

* The idea of a group of operations may be generalised into the more general idea of a group of elements which may include numbers, classes of numbers, forms (homogeneous functions), classes of forms, etc. [Author's note.]

the three straight lines which join the diametrically opposite corners, round the four lines joining the centres of the opposite faces, and round the four lines joining the mid points of the opposite edges. Including in these the rotation which does not change the position of the corners of the octahedron we get in all 24 rotations which bring the octahedron to coincide with itself.

But groups may be formed out of an infinite number of operations (or elements). We might, for example, have the infinite aggregate of all the transformations of the variable x into x' expressed by $x' = x + m$, where m is any real number. This aggregate is a group; for if S is the transformation $x' = x + m$, and S' the transformation $x'' = x' + m'$, then the transformation SS' is evidently $x'' = x + (m + m') = x + m''$ and belongs to the same aggregate.

The number m is called the parameter of the group; and from the time of the celebrated Norwegian geometer Sophus Lie infinite groups are classified according to the number of their parameters. The group that we have examined has one parameter and is denoted G_1 . If n parameters enter into a group of transformations of one or several changes the group is called n fold and is denoted by G_n .

The transformations N , as is easily seen, form a threefold group and therefore may be denoted by the symbol G_3 .

Another mathematical idea, that of invariance, is clearly connected with the idea of a group. A symmetrical function of four letters (for example, $x_1 + x_2 + x_3 + x_4$ or $x_1 x_2 x_3 x_4$) which preserves the same meaning throughout all the 24 permutations of these letters is invariant in relation to this group of all possible transformations. We can subdivide this group of 24 permutations into a group of 8 permutations for which the function $x_1 x_2 + x_3 x_4$ will be invariant.

The function $\sin \frac{2\pi x}{m}$ is invariant for all transformations of the group $x' = x + m$, if m is a whole number. The function $x^2 + y^2$ is invariant for all transformations of the group

$$x' = x \cos \alpha + y \sin \alpha, \quad y' = -x \sin \alpha + y \cos \alpha,$$

because evidently $(x')^2 + (y')^2 = x^2 + y^2$. From this illustration it will be seen that Galileo's principle can be stated as follows:

The equations of motion of a material particle are invariant for all transformations of the three-fold group N .

Analytical mechanics proceed deductively, on the basis of Newton's laws of motion, from the equations of motion of a system of material particles which therefore possess the property of invariance for the group of transformations N .

These general dynamical equations made it possible to complete the edifice of celestial

mechanics, whose foundations and chief results had already been given in the "Principia." The motions of the planets, and all the irregularities in their motions, the irregularities in the motion of the moon, the phenomena of precession and nutation, the shape of the earth, the tides—all were deduced as consequences of the dynamical equations and the law of gravitation. In the middle of the nineteenth century the irregularities in the motion of the planet Uranus led to the discovery of Neptune. By the beginning of the twentieth century all that remained unexplained was the motion of the perihelia of the planets. The most important of these was that of the planet Mercury which amounts to $43''$ in a century. The various hypotheses which were constructed to explain this irregularity could not stand criticism. In chapter v we shall meet with the explanation of this irregularity by means of Einstein's theory of gravitation. Parallel with the development of celestial mechanics there was a development of mathematical physics which used the same dynamical equations to explain physical phenomena. The mechanical picture of the world consisting of atoms moving according to the laws of classical mechanics, for a long time appeared to be the only possible one. Laplace imagined a powerful intelligence who, having written down the "world formula"—a

system of differential equations of the motions of all the atoms of the universe, might by integrating calculate the past and predict the future. In 1872 Emile Dubois-Reymond in a speech* which produced an indelible impression on us, the men of the '70's, recalled these words of Laplace, and applied his "Ignorabimus" only to one problem—the explanation of consciousness. But there had arisen another point of view, the electro-magnetic, due to FARADAY (1809–1867). This regards the matter of atoms only as separate points of the electro-magnetic field, the theory of which is deduced from Maxwell's equations. Nevertheless, classical mechanics could be proud of new successes during the last half century. We need only recall the kinetic theory of gases, and the calculation of the number of molecules based on it, and its application to the solution of the structure of the Milky Way, etc.

3. CRITICISM OF ABSOLUTE SPACE, TIME AND MOTION. GEORGE BERKELEY

Such are the results attained at the present time by the mechanical point of view founded on Newton's laws. But even in his lifetime the results attained by him and published in the "Principia" were so grand that it required great audacity to criticise their fundamental hypo-

* "Ueber die Grenzen der Naturerkenntung."

theses. We know that among these were included the hypotheses of absolute space, time and motion—conceptions which Newton explains under the influence of English Platonism. These hypotheses were criticised by one of the most profound thinkers of all time, a man also inspired by a burning love for humanity. Such was GEORGE BERKELEY (1685–1753), an Irishman by origin, a clergyman and afterwards a bishop of the Church of England.

The whole of his importance in the history of European philosophy has only recently become known. A complete edition of his works, in which for the first time is printed his youthful diary “Commonplace book,” only appeared in 1870. From this we see that before 1705 he was occupied with questions of absolute space and time. Then he worked out his ‘New theory of vision’ (1709) in which the psychological theory of the formation of presentation of space is put out. The development of this theory appears in his ‘immaterialism,’ which reduces all existence to perception (*esse est percipi*) and on this basis denies the existence of matter.

This doctrine—a synthesis of the philosophies of Protagoras and Plato, as it is happily called by Berkeley’s editor, Fraser—is set forth in two of his works in his “Principles of Human Understanding,” which appeared in 1710, and in

his three "Dialogues between Hylas and Philonous," printed in 1713. In both works the criticism of general and abstract ideas occupies a prominent place. From such a critical attitude to abstractions it was natural to pass to the criticism of Newton's hypotheses. This criticism is set out in §§ 110-117 of the "Principles." His separate work, "De Motu" which is Berkeley's answer to the subject set by the Paris Academy, is also devoted to this criticism. It did not get the prize, and Berkeley printed it in 1721. This small book of 25 pages divided in 75 paragraphs still has to-day, in my opinion, more than an historical interest. It deals in logical sequence with three questions—the causes of motion, its nature and the communication of motion.

After declaring his esteem for the results which had been obtained by the hypotheses of the mathematical method for the study of the problems of motion, Berkeley criticises the terms which are used in mechanics and the definition of motion which are far more difficult to understand than motion itself (§ 43).

'It is impossible to separate motion from the space traversed by the moving body and from the time required to traverse this space. Separated from this, motion is an abstraction which does not correspond with reality.'*

* This is not a literal translation of § 43.

Berkeley gives the following rules which he considers useful for the study of the laws of motion :

‘(1) To distinguish between mathematical hypotheses and the natures of things: (2) To beware of abstractions: (3) To consider motion as something sensible or at any rate something imaginable and to be content with relative measurements. If we observe these rules the clearest and most strongly scientific theorem of mechanics will remain inviolable, but the investigation of motion will be freed from a thousand useless subtleties and abstractions’ (§ 66).

The reference to relative measurements undoubtedly refers to Newton’s doctrine of relative motion, and the most interesting part of “De Motu” is the criticism of Newton’s doctrine of absolute space, time and motion (§§ 52–65).

‘Absolute space,’ says Berkeley, ‘is infinite, immobile, indivisible, not perceivable by the senses, unrelated to anything, without distinction between its parts. Thus its attributes are negative, it is mere nothing (*merum nihil*). And what kind of space is it which cannot be divided and which we cannot imagine?’ (§ 53).*

This powerful criticism of the idea of absolute space is completely natural in the thinker who first opposed to the prevailing opinion of European philosophy that space was something objective, the explanation of spatial presentations

* Much abbreviated.

by the association of the palpable sensations of moving and seeing. In his criticism of the idea of space lies the immortal service of the author of "A new theory of vision," "The Principles of Human Understanding" and the "De Motu."

From his criticism of the idea of absolute space Berkeley proceeds to absolute and relative motion.

If every place is relative then every motion is relative and as motion cannot be understood without a determination of its direction which in its turn cannot be understood except in relation to our or some other body. Up, down, right, left, all directions and places are based on some relation and it is necessary to suppose another body distinct from the moving one. . . . so that motion is relative in its nature, it cannot be understood until the bodies are given in relation to which it exists, or generally there cannot be any relation, if there are no terms to be related.

Therefore if we suppose that everything is annihilated except one globe, it would be impossible to imagine any movement of that globe.

Let us imagine two globes and that besides them nothing else material exists, then the motion in a circle of these two globes round their common centre cannot be imagined. But suppose that the heaven of fixed stars was suddenly created and we shall be in a position to imagine the motion of the globes by their relative position to the different parts of the heaven (§§ 58-59).

In §§ 60-62 Berkeley considers Newton's argument in favour of absolute space, based on the experiment of a rotating vessel filled with water, and shews that the motion of the vessel only

appears to be circular, but that in fact it is very far from being circular, because it is necessary to take into consideration not only the movement of the vessel, but also the rotation of the earth about its axis, the monthly revolution of the earth about the centre of gravity of the earth and the moon and the yearly revolution of the earth about the sun. Thus it is impossible to represent absolute motion. And it is unnecessary to introduce it into science; in the place of absolute space we can relate all motions to the fixed stars.

Such were the objections which Berkeley raised to Newton's definitions. The definition of space, as a substance with an objective existence also met with an objection from another quarter, that of Leibniz and his school. For them space and time are ideal orders of things.

'Space and time, extension and motion,' writes Leibniz, 'are not things, but methods of contemplating them (*modi considerandi*).'*

Thus the views of Leibniz were directly opposed to those of Berkeley. The abstractions which appear to Berkeley to be the source of errors, are for Leibniz, on the contrary, the basis of a rational and scientific understanding.

But metaphysical considerations lost their importance in view of the enormous importance

* *Opuscles et Fragments inédits de Leibniz*, par L. Couturat (1903), p. 522.

acquired by Newton's Natural Philosophy. It is understandable that learned men working out mechanical problems by Newton's method should defend his opinions on space and time. These opinions we find in the "Pheronomia" of the first Russian Academician, Hermann, in the works of Maclaurin, the most prominent representative of English mathematical science after Newton, and finally in Euler.

4. THE SUCCESSES OF MATHEMATICAL PHYSICS

The greatest interest attaches to the views of EULER (1707-1783), who on several occasions expressed his opinion on these problems. In his "Mechanics" (1736) Euler defines place as the part of the indivisible, infinite space which contains the corporeal world of that space, which is got as an abstraction from the world of reality. 'We do not assert,' he says, 'that such an infinite space exists, . . . but it seems necessary to postulate something without which we cannot judge whether a body is at rest or in motion.'* But such a point of view in which 'empty space' is merely a hypothetical postulate is inevitably unsatisfactory and requires some foundation. Naturally the question arises 'How can a hypothesis serve as a basis for experimental judgments?' Is it not flagrant contradiction of the

* Chap. I. Section 2 to Definition 2 (abbreviated).

Newtonian 'Hypotheses non fingo' to take a hypothesis as the basis of mechanical science? Euler therefore in 1748 devoted a special memoir ('Réflexions sur l'espace et le temps' in the Memoirs of the Berlin Academy) to prove in detail that the fiction of absolute space can be based on the fundamental laws of mechanics and in particular on the principle of inertia.

'What is the essence of space and time is not important; but what is important is whether they are required for the statement of the law of inertia. If this law can only be fully and clearly explained by introducing the ideas of absolute space and absolute time, then the necessity for these ideas may be taken as proved.'* Euler goes on to prove that the attempt to represent the law of inertia as an empirical law, which applies to bodies on the earth so far as we can observe their motions relative to the fixed stars, cannot withstand criticism. A scientific mechanics cannot depend on the existence or non-existence of the fixed stars and therefore must be constructed on the postulate of absolute space and absolute time. It is not difficult to see (from the reference to the fixed stars) that here Euler is attacking Berkeley.

But apparently Euler himself was not satisfied with his argument, and after seventeen years, in

* This is Cassirer's summary in *Das Erkenntnisproblem*, vol. II, p. 476.

his "Theoria motus corporum solidorum" (1765), he returns to the question of absolute and relative motion.* In the first chapter he insists on the fact that motion and rest, so far as they are accessible to experiment, are always relative. 'Between rest and motion there is no essential difference, for both rest and motion can be attributed at one and the same time to one and the same particle according to what body we refer it.'† But in the second chapter he gives an affirmative answer to the question whether rest and motion can be taken to be attributes of a body independently of its relation to other bodies by enunciating the axiom. 'Every body, without any relation to other bodies, is either at rest or in motion, *i.e.* is either absolutely at rest or has absolute motion.' Euler affirms that the denial of absolute rest and motion not only compels us to renounce the laws of motion, but he also asserts that such laws of motion cannot exist.

This was how Euler considered one of the fundamental questions of mechanics. In the person of Euler against the subjectivism of Berkeley and the rationalism of the school of Leibniz-Wolff mathematical physics came to the defence of the Newtonian position, and after Newton's death (thanks in a considerable degree

* In 1755 Boscovitch had published his *De motu absoluto, an possit a relativo distingui*.

† Cap. 1. Def. 3, Cor. 3.

to the labours of Euler himself) mathematical physics achieved new and powerful results derived from these fundamental hypotheses. In this way there was a dispute between the victorious mathematical theory which rested on shaky epistemological foundations and the other two philosophical points of view already completely developed, but which were mutually contradictory. The solution of the question on which side the truth lay belonged to the theory of knowledge. The problem of the eighteenth century was to support if possible the mathematical theory, so brilliantly justified by experiment and observation, to consider seriously the arguments on both sides, and to find the fraction of truth in each of these contradictory philosophical views of space (Newton, Berkeley and Leibniz).

5. SPACE AND TIME ACCORDING TO KANT

KANT in his youth attacked the solution of this problem, and his reflections on it occupied him till his old age. At first he was under the influence of Leibniz and Wolff, but after that, as he himself says in the Preface to the "Prolegomena," he was awakened from his dogmatic slumber by the philosophy of Hume, which closely resembles that of Berkeley. He speaks of his relation to Berkeley's ideas in the Appendix to the "Prolegomena," contrasting his critical idealism to the

dogmatic idealism of Berkeley.* On the other hand, from his youth up he was under the powerful influence of Newton whom he took as his authority on questions of space, mechanics and cosmology. In his first work, "Gedanken über die wahre Schätzung der lebendigen Kräfte" (1746), written when he was twenty-two, he, with the daring of youth, attacks the problem why space has three dimensions and finds the reason to be that the soul receives impressions in accordance with Newton's law of gravitation, inversely as the square of the distance. In 1755 he published his "Allgemeine Naturgeschichte und Theorie des Himmels," in which he suggested that the planets were formed out of a rotating mass. This hypothesis was afterwards developed by Laplace, and is therefore known as the Kant-Laplace theory. In this book he defends Newton's view that space is prior to all things contained in it which have an objective existence. Nevertheless, in spite of being carried away by the Newtonian philosophy, in the first of the works in which he specially deals with the question whether motion is absolute or relative ("Neuer Lehrbegriff der Bewegung und der Ruhe" (1755)) he decidedly adopts relativism, takes up the definite position that all motion is relative and from this point of view hits upon an

* And the sceptical idealism of Descartes.

inaccurate statement of the law of inertia. 'I cannot say,' he writes, 'that a body moves if I cannot point out the objects in relation to which it changes its position. If I should represent to myself a mathematical space empty of everything only as a possible receptacle for bodies, that would not help me. For how could I distinguish its different parts and consequently different places, if they are not occupied with something corporeal?'* But afterwards he again yields to the authority of Newton and Euler, and in his "Versuch, den Begriff der negativen Grössen in die Weltweisheit einzuführen" (1763) he defends Newton's views on space and time, making considerable use of Euler's arguments.

About this time Kant finally got free from the influence of the philosophy of Leibniz and Wolff and, in opposition to their school, took the view that space had an objective existence. In his interesting memoir (1769), "Von dem ersten Grunde des Unterschiedes der Gegenden im Raume," he makes use of symmetrical bodies to show that absolute space has its own reality not merely independent of the existence of matter, but as furnishing an indispensable condition of its existence. But within two years Kant renounces the idea of an objective space. Our well-

* Hartenstein's Edition of the *Collected Works* (1867), vol. II. p. 17.

known philosopher, Vladimir Sergeivitch Soloviev, in his article on Kant (in Brockhaus and Ephron's encyclopaedic dictionary) attributes this change in Kant's view to the influence of the Swedish mystic Swedenborg* (1668-1772). Kant's "Traüme eines Geistersehers" (1766) is devoted to an analysis of the mystical philosophy of Swedenborg. Kant, who was then under the influence of the English empirical philosophy, was extremely sceptical as to Swedenborg's theories and visions, but one of the fundamental ideas of the Swedish mystic—the doctrine of the ideality of space and time—became in 1770 one of the fundamental ideas of the philosophy of the sage of Königsberg. Schopenhauer, and Cassirer, in his "Erkenntnisproblem," shewed that the doctrine of the ideality of the world had already been expounded by Maupertuis in the year 1752. By this or some other means in his Latin dissertation "De mundi sensibilis et intelligibilis forma atque principiis" (1770) Kant first, says Soloviev, appears as an original thinker with a new and profound view on the subjective character of space and time. This doctrine of space as *a priori*, preceding all experience, a subjective form of our

* Swedenborg's mysticism did not prevent him from being a serious mineralogist. The well-known chemist, Dumas, in his sketch of the philosophy of chemistry, calls him the father of crystallography. [Author's note.]

contemplation constitutes one of the most important doctrines of the "Critique of Pure Reason" which have put their stamp on the whole gnosiology of Kant. In the "Critique" Kant's views on Euclid's axioms and postulates, which the Greeks took as the foundations of Geometry, are of decisive importance. Among these axioms are (1) two straight lines can only have one point in common, (2) a perpendicular and a line inclined to the same straight line always intersect. Kant makes use of the fact that all the axioms of geometry seem to us to be necessarily true, and that we cannot imagine a space to which these axioms do not apply—to shew not only that space is transcendental, a form of pure contemplation independent of experience, but also that Euclid's axioms have a similarly transcendental origin. This is the basis of his "Transcendental Aesthetic."

Neither in his "Critique of Pure Reason" (the first edition appeared in 1781, the second in 1787) nor in his "Prolegomena zu einer jeden künftigen Metaphysik" (1783) does Kant deal specially with the axioms of mechanics and the problems of absolute space, time and motion due to Newton's doctrines. But these questions continued to occupy Kant, and in 1786 he published his "Metaphysische Anfangsgründe der Naturwissenschaften."

This comparatively small work is, on the one hand, an apology for the mechanical view of the universe ('natural science is throughout either a pure or an applied teaching about motion'*) and in particular action at a distance; on the other hand, it is a commentary on the foundations of Newtonian mechanics. In the Preface we find the well-known phrase 'a doctrine of nature can only contain so much science proper as there is in it of applied mathematics'† and two inaccurate prophecies of the impossibility of mathematical chemistry and mathematical psychology. In accordance with the four categories (quantity, quality, relation and modality) the work is divided into four parts in which motion is regarded either as a quantity or as a quality of matter or in its relation to matter or from the point of view of modality (possibility, reality, necessity). In the first part Kant brings forward strong arguments in favour of the view that motion can only be relative. But in the concluding part, by speculative arguments which recall Aristotle's physics, Kant comes to the conclusion that 'absolute space is then necessary not as a conception of a real object but as a mere idea which is to serve as a rule for considering all motion therein as merely relative.'‡ With this

* Bax's translation, 1883, p. 147.

† *Ibid.* p. 141.

‡ *Ibid.* p. 239.

obscure syllogism Kant at the end of his life thought he could reconcile the defence of Newton's position in regard to absolute space and motion with his own doctrine of space, which involves a fundamental denial of those metaphysical views of Newton on which the "Principia" is based.

In view of the importance which Kant's doctrine of space has had not only on the foundations of Geometry, about which we will speak in the next chapter, but also on the foundations of mechanics, it is useful to pause on that doctrine, the more so as it brings out the synthetic reconciling nature of the Kantian philosophy.

In the previous chapter we saw that in the ancient Greek Philosophy in general (not only of Plato and Democritus, but also of Aristotle) space was recognised as something which had an absolute existence. Perhaps Protagoras was an exception, but this we can only surmise. This view of space as something real, independent of man, passed over into modern philosophy. We find it in Descartes, Newton and the Atomists (Gassendi and Hobbes). In detail they differed. For Descartes space is identical with matter, for Newton and the Atomists it embraces everything and is as it were opposed to matter. In the English Platonists it appears as a spiritual substance. But modern philosophy has produced two other

views of space. It was Berkeley's immortal service that he decidedly rejected the external reality of space and recognised it exclusively as the subjective result of sensations of sight, touch and movement.

For Leibniz space is a subjective perception, but it consists of the objective order of the existence of 'things in themselves.' Leibniz' view is essentially different from Berkeley's, for whom 'things in themselves' did not exist.

Kant's object was to reconcile all these points of view. Space is something which envelops everything (Newton), but this something has no objective existence and is merely a form of our contemplation. Thus Kant asserts, with Berkeley and Leibniz, that space is subjective, but he does not agree with Berkeley. Space is not the result of sensations of sight and touch, it is prior to all experience. He does not agree with Leibniz, for we have no right to assert that the order of the relations to the reality which is inaccessible to us, or to 'things in themselves' corresponds to the relations of our space and time. Just as Kant, in his doctrine of space, broke with the one-sided views of the philosophers who preceded him, so all his philosophy was a profound synthesis of the former philosophical systems. As we have already mentioned, his "Principles of Metaphysics" is an apology for the mechanical view of the

world. The world of phenomena is subordinate to the unshakeable laws of necessity. But at the same time, the recognition of free will leads to a fundamental distinction between the physical and the psychical, which does not allow the possibility of explaining the 'consciousness' of the movements of atoms. That part of his doctrine which deals with the relation between thought and experience has a similar character. Helmholtz, in his celebrated speech on "Thought in medicine," describes Kant's relation to these questions as follows: 'It was as clear to him [Kant] as to Socrates that all metaphysical systems which up to that time had been propounded were tissues of false conclusions. . . . But geometry seemed to him to do something which metaphysics was striving after; and hence geometrical axioms, which he looked upon as *a priori* principles antecedent to all experience, he held to be given by transcendental intuition or as the inherent form of all external intuition. Since that time pure *a priori* intuition has been the anchoring ground of metaphysicians. It is even more convenient than pure thought, because everything can be heaped on it without going into chains of reasoning which might be capable of proof or regulation.'

* *Popular Lectures*, 2nd Series, 1881.

6. THE VIEWS OF HELMHOLTZ

The synthetic character of Kant's philosophy is shown in a remarkable degree by the enormous influence which it had and continues to have on human thought. Certainly it is particularly strong in German philosophy; in Anglo-Saxon countries it has to reckon with the influence of the English empirical philosophy, with Berkeley and Hume.*

In Russia, as is justly pointed out by Professor T. S. Masaryk in his excellent book "*Zur russischen Geschichts- und Religionsphilosophie Soziologische Skizzen*" (2 vols. 1913),† the influence of the Kantian philosophy was comparatively unimportant. Among those who are no more, only Soloviev and Lavrov were markedly influenced by it.‡ The influence of those philosophical schools which attributed greater importance to Kant's idealism than to his criticism was much greater. Odoevski, Bakounin, Hertzen and Chernoushevski were attracted by the

* Cf. Lyon's interesting book, *L'Idéalisme en Angleterre au XVIII^e siècle*, Paris, 1882. The most powerful American metaphysicians of the eighteenth century, Samuel Johnson and Jonathan Edwards, were immaterialists; under the latter's influence in Princetown University *The Dialogues of Hylas and Philonous* was used as a text-book. [Author's note.]

† President Masaryk's book has now been translated under the title of "The Spirit of Russia."

‡ Later, Plekhanov, Lenin held metaphysical views on space and time. [Author's note.]

teachings of Fichte, Schelling, Hegel and the Hegelians of the left.

Their doctrines had also great influence in Germany in the nineteenth century. In them Kant's Transcendental Aesthetics either were admitted completely or developed further in a metaphysical direction. In general, Kant's views on space and time were in the nineteenth century mostly diffused among the philosophers and scientists with some exceptions (the mathematicians Gauss, Lobachevski and Riemann, of whom we shall speak in the next chapter). Starting from the fundamental premises of classical mechanics and considering their philosophical basis to be unshakeable, those scientists who were occupied with problems of mathematical physics, strove to extend the mechanical view of the world to all phenomena. The French mathematicians (Cauchy, Poisson, Navier) adapted the equations of the dynamics to explain the theory of wave motion in ether. The Englishmen (Kelvin, Maxwell) tried to make models of the ether and to give a dynamical explanation of electromagnetic phenomena. Helmholtz, in his well-known pamphlet "On the conservation of force" (1848), regarded the laws of the conservation of force as consequences of the central forces of physics, *i.e.* of classical mechanics.

But work on the physiology of the organs of

the senses, chiefly on that very question of vision which formed the starting-point of Berkeley's phenomenological views, led Helmholtz first of all to a decided attack on the metaphysical views of the post-Kantian school.

He was one of the first to take as his motto 'Back to Kant.' Later (we shall deal in greater detail with this in the next chapter), he referred critically to Kant's teaching about the transcendentalism of the axioms, and recognised with Kant that space was transcendental. But Helmholtz did not touch on the basis of mechanics.

As far as we know, the only serious work which appeared before 1872 on the foundations of mechanics and devoted special attention to absolute and relative motion and the law of inertia, takes the same point of view as that which Kant developed in his "Metaphysical Principles." We refer to Karl Neumann's "The principles of the Galileo-Newton theory" (1870).

Neumann perceives that the usual exposition of the principles of mechanics is vague and difficult. He acknowledges that Galileo's statement that a material particle left to itself moves in a straight line is a mere assertion 'suspended in the air' so long as the system is not given with reference to which the direction is defined. The law of inertia must have a foundation. Neumann, as a way out of the difficulty, proposes to regard

all motion as absolute defined by means of a hypothetical body Alpha. It is only such a hypothetical body as can serve for our judgment of direction. We must conceive all motion as in relation to it. In this theory we easily recognise a return to the views of Damascius and Simplicius in their commentaries on Aristotle which were opposed later on by the majority of the scholastics of the school of Averroes.

7. THE EMPIRIOCITICISM OF ERNST MACH

At the same time as Karl Neumann, Ernst MACH (1838-1916) turned his attention to the difficulties connected with the law of inertia.

Mach often made his readers acquainted with the process of thought which led him to the 'phenomenological point of view,' which for us mathematicians and physicists is inseparably connected with his name. He is always near to that 'empiriocriticism or empiriomonism' at which Avenarius and Schuppe arrived, not from a physical or physiological, but from an exclusively gnosiological point of view. To understand Mach's point of view it is really extremely useful to be acquainted with its development. 'My naïvely realistic view of the world,' he writes, 'was shaken by reading Kant's "Prolegomena" in 1853'; that is, when he was fifteen. It gave a shock to the critical thinking of the young man.

But he quickly recognised the 'thing in itself' as natural truth and an instinctive but idle illusion. 'I turned then,' he continues, 'to Berkeley's point of view secretly preserved by Kant and to Hume's view of causation.' The thought of Berkeley and Hume is undoubtedly more logical than that of Kant—such was Mach's conviction even in his early youth. This conviction, which clearly separates him from the overwhelming majority of physicists and naturalists (including Helmholtz), he never abandoned. 'I am convinced,' he wrote in 1910,* 'that Kant's philosophy is a marked step backwards in comparison with Berkeley and Hume.'

Undoubtedly the influence of Berkeley and Hume, and an acquaintance with the mathematical psychology of Herbart and the psychophysics of Fechner, and also the contemporary study of physics and even more of the physiology of the organs of sensation, led him at the end of the sixties to working out his 'antimetaphysical' point of view, as he called it. But in his fundamental view of the object of physics—to establish the connection between phenomena—Mach comes too near to Auguste Comte. But at that time, as for Comte, physics was every-

* See the Author's "Les idées d'Auguste Comte sur la philosophie des mathématiques" (*Enseignement mathématique*, 1898).

thing and psychology—nothing. Mach regards the physical and the psychical as equally important sources of our knowledge. The simplest ‘elements’ of these two (feelings and physical properties) are identical in themselves and are distinguished only by the way in which they are regarded. In close connection with his general outlook on the world and his view of the aim of physics, is his critical relation to the mechanical view of nature which he had already definitely formulated in 1872 in his well-known work “History and Root of the Principle of the Conservation of Energy.” ‘I believe that I have shown that one can hold, treasure, and also turn to good account the results of modern natural science without being a supporter of the mechanical view of nature, and that this conception is not necessary for the knowledge of the phenomena and can be replaced just as well by another theory, and that the mechanical conceptions can even be a hindrance to the knowledge of phenomena.’* In an interesting polemic with Planck in 1910 Mach defended the same idea of the necessity of constructing physics without any artificial hypothesis. He did not share the belief of the physicists that ‘atoms are no less real than heavenly bodies, that it is just as credible that an atom of hydrogen weighs 1.6×10^{-24} grammes

* English translation, p. 54.

as that the moon weighs 7×10^{25} grammes.' To-day, after ten years, the dispute has been decided in favour of Planck. But, in our opinion, the proofs of Perrin, Rutherford and others of the granular nature of matter are not a decisive argument against the phenomenological point of view.

With his general critical attitude to the mechanical point of view, it is natural that Mach should pay special attention to the fundamental postulates of the classical Newtonian mechanics and mainly to Newton's definitions of mass and the law of inertia.

Keeping within the limits of our work, we only pause on Mach's attitude towards the law of inertia and quote *in extenso* those passages in which he formulates his attitude to this law and to the allied questions of absolute and relative motion. In his work of 1872 he writes: 'Obviously it does not matter if we think of the earth as turning round on its axis, or at rest while the celestial bodies revolve round it. Geometrically these are exactly the same case of a relative rotation of the earth and the celestial bodies with respect to one another. Only the first representation is astronomically more convenient and simple. But if we think of the earth at rest and the other celestial bodies revolving round it, there is no flattening of the earth, no Foucault's experi-

ment, and so on—at least according to our usual conception of the law of inertia. Now one can solve the difficulty in two ways. Either all motion is absolute, or our law of inertia is wrongly expressed. Neumann preferred the first supposition, I, the second. The law of inertia must be so conceived that exactly the same thing results from the second supposition as from the first. By this it will be evident that in its expression, regard must be paid to the masses of the universe.....All bodies, each with its share, are of importance for the law of inertia.’*

We see, how, already in 1872 Mach explains inertia and centrifugal force by that hypothesis of which we find a hint in Berkeley and which Euler called ‘very strange and contradictory to the dogmas of metaphysics,’ at the same time recognising that it was very difficult to refute it—the hypothesis of the influence of the fixed stars or the mass of the world space.

Later, in his mechanics, the first edition of which appeared in 1883, Mach formulated his ideas in greater detail. ‘For me only relative motions exist, and I can see, in this regard no distinction between rotation and translation. When a body moves relatively to the fixed stars, centrifugal forces are produced; when it moves relatively to some different body and not related

* English translation, p. 76.

to the fixed stars, no centrifugal forces are produced. I have no objection to calling the first rotation so long as it be remembered that nothing is meant except relative rotation with respect to the fixed stars.*

Mach, like Berkeley, does not consider Newton's experiment with the rotating vessel to be convincing. It only shews that the relative rotation of the water in relation to the walls of the vessel does not arouse perceptible centrifugal forces, but that the latter arouses relative rotation in relation to the mass of the earth and the remaining heavenly bodies. On the one hand, nobody can tell what would happen if the walls of the vessel got thicker and more massive, till finally they were some miles thick. On the other hand, could we keep Newton's vessel of water immovable and compel the fixed stars to revolve and then prove that there is no centrifugal force?

This experiment is unrealisable, there is not even any sense in thinking about it. So then Newton's distinction between rotation and translation is an illusion and the idea of absolute motion is 'absurd, empty and useless.'

But these views of Mach, not only in 1872 when he published "Principles of the Conservation of Energy," but in 1883, when the first edition of his "Mechanics" appeared, seemed to the majority

* English edition (1919), p. 542.

of learned men to be either paradoxical or a sign of 'confusion of physical ideas,' as Planck in 1911 expressed it about the relativity of rotation.

But each decade brought Mach new adherents. In the aims which Kirchhoff attributed to mechanics of 'a complete and the simplest description of motion'* and in Duhem's "La théorie physique," Mach had a right to see an agreement with his views on the economical significance of science and his negative attitude to hypothetical physics. Particularly in the nineties, when under the works of the mathematicians on non-Euclidean Geometry, the theory of hypercomplex numbers, and the theory of magnitudes strengthened the interest in the theory of knowledge and in particular of the axioms on which science is based. Mach's opinions on the axioms of mechanics were subjected to serious consideration and met with the complete sympathy of many learned men and thinkers. In the works of Stallo, Pearson, and Clifford and in Poincaré's brilliant books, full of profound and illuminating ideas, there were developed views on the theory of knowledge and particularly on the axioms of mechanics which coincide with many of Mach's opinions. In 1904, in the sixth edition of his "Mechanics," Mach already announced, as he counted

* *Vorlesungen über mathematische Physik. Mechanik.* (1874), p. 1.

up the number of his adherents, that 'the number of decided relativists who deny the barely intelligible hypothesis of absolute space and time is growing rapidly and soon there will not be one prominent partisan of the contrary opinion.' Then he dreamed of an ideal mechanics based on a principle from which both accelerated motion and motion due to inertia would equally flow.

Einstein's general theory of Relativity is based on epistemological premises which, as we shall see in the fifth chapter, coincide with the ideas of Mach. But in order that it might be constructed and accepted with the sympathy of the majority of the prominent scientists and thinkers of our time it required two things: *First* that the evolution of ideas about space should change the prevailing view of the relation between geometry and physics, between space and the phenomena taking place in it (the view of space, to use Weyl's clever expression, as a hired barracks); *Secondly*, instead of a three-dimensional space and a one-dimensional time, there should be worked out a general idea of the world, an aggregate of events in a manifold of four dimensions. The first became possible through the successes of non-Euclidean geometry, the second through the development of our physical experiments. The next two chapters will be devoted to these two questions.

CHAPTER III

NON-EUCLIDEAN GEOMETRY

Lobatchefsky a laissé en géometrie
une trace glorieuse, impérissable.

(Telegram from the professors of the Sorbonne to
the University of Kazan on 23rd October 1893).

I. CRITICISM OF EUCLIDEAN GEOMETRY

Non-Euclidean Geometry, whose philosophical importance is denied by no one at the present day, and whose great importance for natural science appears in the general Theory of Relativity, is a result of the characteristic strivings of the human mind not only to raise the edifice of thought higher and higher, but to establish it on unshakeable foundations.

This striving after a basis for the geometrical knowledge acquired practically from land surveying and architecture, by the measurement of the fields in the valley of the Nile, the construction of altars and palaces in the valleys of the Ganges and the Euphrates, gave that well arranged system, which is expounded in Euclid's "Elements," and forms one, and perhaps the most important, of the services rendered by the genius of the Greeks.

It is almost certain that Euclid had predecessors among whom were the Geometers of the Pythagorean school. The connection of Euclid's "Elements" with the Pythagorean doctrines is

seen from the fact that it ends with the theorem about regular polyhedra and from the fact that the arithmetical books of the "Elements" finish with the theorem about perfect numbers.* Paul Tannery considers it indubitable that already in the Pythagorean school an attempt had been made to create a system of Geometry in the book which bears the title "The tradition of Pythagoras." Zeno's dialectics, the critical sensualism of Protagoras who we know attacked the mathematicians, made the latter more and more rigorous in their proofs and caused them to complete their fundamental axioms and definitions. In this way Euclid's "Elements," which outlived in importance Aristotle's "Physics" and Plato's dialogues, came into being as the result of the work of many generations of mathematicians. Euclid proved to be an excellent school for the development of the mind and a necessity for schoolmasters. But modern criticism found many flaws in it. One of them—the theory of parallel lines—was already noticed by the critical genius of the Greeks. 'It is easy to prove,' writes Lobachevski in the introduction to "New Elements of Geometry with the complete theory of parallel lines," 'that two straight lines inclined at the same angle to a third never meet; Euclid assumed the converse—that two straight lines not similarly inclined to

* Book ix. Prop. xxxii.

a third must always intersect; various methods were adopted to prove the accuracy of the latter assumption; among such methods were comparing the infinite planes in the opening of angles and between perpendiculars—admitting the dependence of the angles only on the contents of the sides,—or finally giving new properties to a straight line to supplement the definition. Some of these proofs may be called clever, but all in general are false, insufficient in their foundations and without the necessary rigour.'

A detailed systematic history of the attempts to get a basis for the theory of parallel lines has not yet been written, but the interesting book of Professors Engel and Stäckel "*Theorie der Parallelinien*" (1895) contains rich material for the history of the question.

Proclus, a commentator of the fifth century, had already pointed out that it was possible that at a point outside a straight line there might be two 'asymptotes' to it, one to the right, the other to the left. At the beginning of the seventeenth century a chair of Euclid was founded at Oxford. Wallis, one of the first professors to occupy this chair, replaced Euclid's postulate by the postulate that 'that to every figure there corresponds a similar figure of any size.' In the eighteenth century the Jesuit Saccheri, in his remarkable book "*Euclides ab omni naevo vindicatus*,"

pointed out three possibilities, which he called the right angled, acute angled and obtuse angled hypotheses. The investigations of Lambert, Legendre, and Taurinus led to many remarkable results, but they all stopped half way, convinced that the 'flaw' in the theory of parallels was only a casual flaw and could be got rid of. It is indubitable that in 1799 GAUSS had already found some of the important postulates of that non-Euclidean geometry which to-day is called Lobachevski's. But he never published his conclusions during his life as he 'feared the outcry of the Boeotians.'* Only N. I. Lobachevski and Johann Bolyai were daring enough to do it. After prolonged attempts to prove Euclid's postulates, LOBACHEVSKI on the 12th February 1826, at the public sitting of the physico-mathematical faculty of the University of Kazan, expressed his conviction that 'these ideas (of geometry) do not contain the truth which they wished to prove and which can only be verified like other physical laws by experiments, for example by astronomical observations.'

The road which both Lobachevski and Bolyai took was that of synthetic geometry. But from the time of Descartes' "Geometry" (1637) geometry had acquired for the solution of its problems a new and powerful weapon in algebra

* Letter to Bessel, 27 January 1829.

and the differential calculus. The study of problems about the shape and position of curves, surfaces and bodies had led to the solution of problems of numbers and functions. After Descartes himself had shewn the importance of the new method for the study of plane curves, Euler, Meusnier and Monge applied the method of co-ordinates, the idea of which had already appeared in the fourteenth century (Nicholas Oresme, 1320–1382) to the study of curves in space (one-dimensional manifolds) and surfaces (two-dimensional manifolds). The great step in advance, the significance of which is only appreciated thanks to the general Theory of Relativity, was taken by Gauss in 1828 in his “*Disquisitiones generales circa superficies curvas*” and consists in what may be termed the ‘internal theory of surfaces.’ We will pause on the fundamental ideas of the Gaussian theory. The position of every point on the globe, as we know from our childhood, is defined by latitude and longitude, that is, the intersection of some parallel circle with some meridian. Generalising this definition of a place on a sphere for surfaces in general, Gauss introduced curvilinear (to-day we call them, after Einstein, ‘Gaussian’) coordinates into the theory of surfaces. On the surface there is drawn first a system of curves u , each of which corresponds to a definite numerical value of the parameter u .

No one of the u curves ought to intersect another one and only one curve may be drawn through each point of the surface. On the same surface another system of curves v is drawn, which satisfy the same conditions and to each of which corresponds one value of u and one value of v . These values are the Gaussian coordinates of the point. The equations of the surface can be expressed by $x = \phi(u, v)$, $y = \psi(u, v)$, $z = \chi(u, v)$ where x, y, z are the Cartesian coordinates of the point. Two infinitely near points on the surface whose Cartesian coordinates are x, y, z and $x + dx, y + dy, z + dz$ will have as Gaussian coordinates u, v and $u + du, v + dv$. The line element of the surface (the distance between these two points) which in Cartesian coordinates is $dx^2 + dy^2 + dz^2$ in Gaussian coordinates has the form

$$g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2$$

i.e. a quadratic function of du and dv .

Gauss called these coefficients g_{11} , g_{12} , g_{22} fundamental quantities of the first order to shew that these three numbers, or, to express it differently, the corresponding quadratic form defines the internal metrical relations which characterise the surface entirely independently of its position in space.

As Gauss shewed, the length of curves drawn on the surface—all the *metrical* properties of the

surface—depend exclusively on the quadratic form. For this reason the quadratic form may be called the *metrical* form of the surface. If the surface is bent, that is, if its appearance is changed so that all lines on the surface conserve their length, the expression for the element of length remains unchanged.

So, for example, for surfaces which can be unrolled on to a plane without any extension or compression, as the cylinder, the cone and the helicoid (screw surface), the line element can always be expressed by a suitable choice of curvilinear coordinates in the form $ds^2 = du^2 + dv^2$ which, on the basis of Pythagoras' theorem, is characteristic of a plane. On the other hand, the line element of a sphere of unit radius, which has the form $du^2 + \sin^2 u \, dv^2$, where u and v are the latitude and longitude of the point, cannot be reduced to the form $du^2 + dv^2$ by any transformation of curvilinear coordinates. But for all surfaces which can be wrapped on a sphere without extension or compression the line element can be represented in the form

$$du^2 + \sin^2 u \, dv^2$$

On the pseudosphere, that is, the surface formed by the rotation of the tractrix, and on all surfaces which can be unwrapped on it the metrical form is given by $du^2 + \sinh^2 u \, dv^2$.

It follows from the constancy of the line element that when the surfaces are bent the geodesic lines (lines of shortest distance) remain geodesics. Thus if a plane surface is wrapped on to a cylinder the geodesic lines of the plane (straight lines) coincide with the geodesic lines of the cylinder (screw lines).

From Gauss's result it follows that all surfaces which can be wrapped one on the other have one and the same geometry. Thus, for example, the geometry of the plane and that of the cylinder coincide. The sum of the angles in a triangle formed of geodesic lines equals two right angles. On the other hand, both for a sphere and for all surfaces which can be wrapped on a sphere the sum of the angles of a geodesic triangle is greater than two right angles, while for a pseudosphere and for all surfaces which can be wrapped on it the sum is less than two right angles.

Finally Gauss introduced a new idea of the measure of *curvature* of surfaces, analogous to the curvature of a curve and shewed that the mathematical expression for this measure of curvature depended solely on the coefficients of the metrical form (g_{11} , g_{12} , g_{22}) and their first and second differentials. From this it follows that for all surfaces obtained by bending one from the other the measure of curvature at corresponding points of the surfaces will be equal. For example,

since the measure of curvature at all points of a plane surface is zero, the measure of curvature is zero for all surfaces on which it can be wrapped. For a sphere of radius R this measure of curvature, which is called the Gaussian curva-

ture, is equal to $\frac{1}{R^2}$ and the Gaussian curvature

has the same magnitude at all points of surfaces which can be wrapped on the sphere. For surfaces which can be wrapped in a pseudosphere, as well as for the pseudosphere itself, the curvature is the same quantity but with a negative

sign $-\frac{1}{R^2}$. In general the curvature of a surface

is a variable quantity. For each point of the surface it is characterised by one number (as temperature is, for instance) and is called a *scalar*. The Gaussian curvature of a surface is a scalar quantity.

2. NON-EUCLIDEAN GEOMETRY

From the standpoint of Gauss's theory of surfaces, Lobachevski's planimetry is the geometry of a surface with a constant negative curvature, *i.e.* it exists, for example, on the surface obtained by the rotation of a tractrix. It is interesting that simultaneously with Lobachevski, Professor Minding of Dorpat University (1840), in studying the geometry of surfaces with a constant negative

curvature, obtained the same formulae as those of Lobachevski's geometry. This remarkable coincidence was not noticed either by Minding or Lobachevski, and only in 1868 was attention called to it by Beltrami in his celebrated "*Saggio di Interpretazione della Geometria non-Euclidea.*" Hilbert shewed that in Euclidean space it is impossible without particular points and lines to represent surfaces which would realise a whole plane surface of Lobachevski.

But if Lobachevski's planimetry can be, though not completely, realised on surfaces in our space, yet its stereometry, which is very different from that of Euclid, gives rise to the question: What properties does the space of our experience possess? If, in addition to Euclid's stereometry, another logically irreproachable stereometry is possible, then what is the basis of the conviction formulated by Kant in his critique (see previous chapter) that Euclid's axioms are the necessary presuppositions of our inner contemplation and consequently of all experience?

Lobachevski understood the connection of this question with astronomy ('the truth of Euclid's Geometry might be verified experimentally for example by astronomical observations') and physics. The following phrases introduced into the "*New Principles of Geometry*" are specially characteristic—'in our mind there cannot be any contradiction if we suppose that

some natural forces follow one, others their own special geometry. . . . Anyhow suppose this a mere supposition for whose support it is necessary to seek out other convincing proofs, but nevertheless it is impossible to doubt that the forces all produce everything—motion, velocity, time, mass—even *distance* and *angle*. All is in the closest connection with the forces.'

Only now, almost 100 years later, can we understand the full depth of the thought contained in these apparently paradoxical words.

But Lobachevski's genius could not perceive everything. Saccheri, as has been mentioned above, had pointed out three possibilities, which he called the hypothesis of the acute, right and obtuse angles. Lobachevski's pangeometry forms the first of these. On the obtuse angled hypothesis a logical system of two- or three-dimensional geometry may be constructed by rejecting the axiom that two straight lines intersect in only one point. But if, as has already been shewn, to straight lines in a plane geodesic lines on other surfaces correspond (on a sphere they are the arcs of great circles), the two-dimensional geometry based on the hypothesis of the obtuse angle is nothing else than the spherical geometry already worked out by the Greek geometers Theodosius, Menelaus, Hipparchus and Ptolemy. The generalisation of this idea to three-dimen-

sional geometry, of a space which, like a sphere, is not infinite but unbounded, a space in which, as in Lobachevski's space, there are no similar figures, but in which the sum of the angles of a triangle formed of geodesic lines is greater than two right angles—is due to a pupil of Gauss, BERNHARD RIEMANN (1826–1866). But the importance of his paper, “Ueber die Hypothesen welche der Geometrie zu Grunde liegen” (written by Riemann in 1854 but only published after his death in 1868), is not limited to the idea of spherical space, which is called Riemann's space. The fundamental idea of the paper is that space, considered as a subject for mathematical study, is only a particular case of a more general idea of a manifold (Mannigfaltigkeit) of n dimensions, an aggregate of elements each of which is defined by n numbers. The idea that our (Euclidean) space is only a particular instance of a more general idea was first expressed by Kant in 1747 in his “Thoughts on the accurate definition of vis viva.” ‘A science of all these possible aspects of space,’ says Kant, ‘would doubtless be the highest form of geometry which the human mind could conceive.’*

Riemann mentions the influence exercised on him by those ideas ‘of the general abstract study

* *Gedanken von der wahren Schätzung der lebendigen Kräfte*
Erster Hauptstück, § 10.

of magnitude whose object consists in the combination of magnitudes, which satisfy the condition of continuity' at which Gauss hinted in some of his memoirs (on the theory of biquadratic residues and the proofs of the fundamental theorems of higher algebra) and the influence of the mathematical psychology of Herbart.* But Riemann apparently was not acquainted with the book which had appeared ten years before he gave his paper, namely *Hermann Grassmann's* "Ausdehnungslehre" (1844). Grassmann's fate in many respects reminds us of Lobachevski's; both of them developed their ideas in small provincial towns (Stettin and Kazan) and for a long time their works did not attract the attention of mathematicians. GRASSMANN (1804-1877), however, was more fortunate than Lobachevski; he did, in extreme old age, live to see the importance of his works recognised. In these works Grassmann develops the doctrine of an extension of any number of dimensions; that is, an extension derived from one and the same element by certain laws of transformation. Geometry or knowledge about space is a particular instance of the knowledge about extension. In it the elements are points, its continuous change is change of position or motion,

* See "On the hypotheses which lie at the basis of Geometry," sect. 1. Clifford's *Mathematical Papers*, p. 56.

and its different conditions the different positions of points in space.

As has been said above, the fundamental idea of Riemann's paper consists in the fact that Riemann considers geometry as a particular case of the more general notion of a continuous manifold of n dimensions, a system of elements (called points) each of which is defined by n numbers (called coordinates) and takes this more general notion as subject for mathematical investigation. In this investigation those measure-determinations (Massbestimmungen) which characterise it are studied.

In this way we have for manifolds of n dimensions the problem which Gauss posed and solved in 1828 for surfaces in a space of three dimensions. In his investigation Riemann makes two very important postulates. The first, the importance of which the talented German mathematician *Weyl* has recently called attention to, consists in the assumption that the length of a vector is independent of its position. In other words, Riemannian Geometry is based on the assumption that every translation of a vector leaves its length unchanged. If a vector is translated from P to P' the length at P' is independent of the route taken. We shall see later that in *Weyl's* generalised Geometry it is not possible to compare lengths (except zero lengths) at dif-

ferent places because the result will depend on the route taken in bringing the two lengths in juxtaposition. That is to say, that the *gauge* system varies from point to point, and we have a special unit of measure at each point. The second consists in the fact that for his line element in the theory of manifolds Riemann takes a generalisation of the element of an arc on a surface. If for a surface, which is a two-dimensional manifold, we have

$$ds^2 = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2$$

then for an n -dimensional manifold, in which each element is defined by n numbers

$$x_1, x_2, x_3 \dots x_n$$

the square of the distance between the element or point $(x_1, x_2, x_3 \dots x_n)$ and the infinitely near element $(x_1 + dx_1, x_2 + dx_2 \dots x_n + dx_n)$ will be expressed by the formula

$$ds^2 = g_{11} dx_1^2 + g_{22} dx_2^2 + \dots$$

$$g_{nn} dx_n^2 + 2g_{12} dx_1 dx_2 + 2g_{13} dx_1 dx_3 + \dots$$

or more shortly

$$\Sigma g_{ik} dx_i dx_k \quad (i, k = 1, 2, 3 \dots n).$$

On these postulates Riemann's differential geometry is founded. Two questions particularly occupied Riemann. First he considered how a manifold could be called flat, *i.e.* could be considered as corresponding to a plane surface in the theory of two-dimensional manifolds or

Euclidean space in the theory of three-dimensional manifolds. In other words, how the line element could be represented in the form

$$ds^2 = dx_1^2 + dx_2^2 + \dots + dx_n^2$$

The investigation of this question leads Riemann to the notion of the curvature of a manifold. But this curvature is sharply distinguished from the Gaussian curvature of surfaces. While the latter is a scalar (that is, it depends only on the position of the point of the surface and is expressed by one number), the Riemann curvature is a so-called *tensor* (that is to say, it is expressed by many numbers, as in the case of the pressure inside an elastic body, or the stresses in a solid or in a viscous liquid). Levi-Civita and Weyl have introduced the notion of parallel displacement. A vector carried round a circuit is, in general, not in its original direction when it has returned to its starting point. This difference is expressed by a tensor of the fourth rank called the Riemann-Christoffel tensor. It is a necessary and sufficient condition for a manifold to be *flat* that this tensor should vanish. Since the tensor is symmetrical it has only $\frac{n^2(n^2 - 1)}{12}$ components,

i.e. 20 if $n = 4$. In flat manifolds a figure can be transferred freely without distortion. But flat manifolds are not the only ones in which such transference is possible. The same property is

possessed by manifolds whose curvature has a constant value other than zero at all points and in every direction from each point.

These general results Riemann attributes to the space which is studied in Geometry. 'Its unboundedness has a greater empirical credibility than any other doctrine of our external experience.'* But this does not imply that it is infinite. Just as a circle has a finite length, although it has neither beginning nor end, just as the surface of the earth on which movement can continue without limit, has nevertheless a finite number of square kilometres, so curved space, however small its curvature may be, may have a finite volume, and will be unbounded. Such a space with a constant curvature is called Riemann's space. Whether our space is of this kind or whether its curvature is zero and Euclidean geometry applies can only be decided by experiment. According to Riemann, the latter is the result of astronomical measurements. Great interest attaches to the question whether Euclid's postulates apply to infinitesimals, and this closely connected with the question of the internal basis of the metrical determination of space: 'If in the case of a discrete (discontinuous) manifold the basis for its metrical determination is comprised in the very idea of this manifold, then for a con-

* Clifford's *Mathematical Papers*, p. 67.

tinuous one it ought to come from without. Therefore the reality which lies at the basis of space, or either constitutes a discrete manifold or the basis of metrical definition, must be sought outside the manifold in the forces which act on it and hold it together.'*

This rather obscure passage nevertheless leaves no doubt that Riemann saw the connection between the metrical relations existing in space and the forces acting on it, just as Lobachevski thought it possible to assert that distance and angles are produced by forces.

3. EXTENSION AND MOTION

For a long time the ideas of Lobachevski and Riemann were not properly valued. To appreciate them requires a philosophical turn of mind which is comparatively rarely met with in specialists. But the talented English mathematician, W. K. CLIFFORD (1845-1879), was imbued with just such a philosophical disposition. He was passionately interested in epistemology. The similarity of his philosophy, which may be described as an idealistic monism, with that of Mach, we have already pointed out at the end of the previous chapter. Clifford's 'mind stuff' is very like the 'world elements' of Mach and Avenarius. In

* Clifford's *Mathematical Papers*, p. 69.

geometry he has left behind some beautiful works, the most important of which is devoted to the study of the properties of elliptic space.

Clifford was a vigorous propagandist of the ideas of Lobachevski and Riemann. He translated Riemann's paper into English and attached so much importance to this translation that he included it in the collection of his own original works.* He called Lobachevski the Copernicus of Russia. But he did not merely diffuse the ideas of the Russian and German mathematicians. He appropriated and developed the idea of the intimate connection between geometry and physics which Lobachevski and Riemann had only hinted at. More than once in his articles and books he expressed the thought that so-called physical changes were due to changes in the curvature of our space. 'We can take as a postulate, that that part of space relatively to which we have knowledge in practice, is homoloidal (that is, Euclidean geometry applies to it), but we have no right to extend this postulate to all space, nor have we any right to deny that the curvature may change in time.' 'Does it not turn out to be the case that all or some of those causes which we call physical arise from the geometrical properties of our space.'

* On this translation see Stallo, *Modern Physics*, footnote to p. 253.

In another passage he formulates his hypothesis as follows:

‘(1) That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely that the ordinary laws of geometry are not valid in them.

‘(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

‘(3) That this variation of the curvature of space is what really happens in that phenomenon which we call *motion* of matter, whether ponderable or ethereal.

‘(4) That in the physical world nothing else takes place but in this variation, subject (possibly) to the law of continuity.’*

When in my speech in memory of N. I. Lobachevski (22nd October 1893)† I quoted Clifford’s hypothesis, I did not, I remember, do this without hesitation; it seemed a brilliant paradox and nothing more. I could not foresee that I should live to see the time when Clifford’s fantastic theory would be regarded as a provision of the mathematical doctrine that the

* “On the space theory of Matter.” *Proceedings of the Cambridge Philosophical Society*, II. 1876. An abstract is printed in Clifford’s *Mathematical Papers*.

† This speech has been translated by Halsted.

metrical properties of space are connected in the closest manner with physical phenomena.

There is another profound thought of Lobachevski. 'In nature,' he says, 'we recognise properly only motions without which sense impressions are impossible. All other ideas, for example geometrical ones, are the artificial products of our mind, derived from the properties of motion and therefore space *per se* has no separate existence for us.'*

More than a century before Lobachevski wrote these words, Berkeley in 1734 in "The Analyst" subjected the foundations of the theory of fluxions to a criticism, which was not always well founded, and developed the idea of 'compensation of errors' which later Carnot took as the basis of his "Réflexions sur la métaphysique du calcul infini" (1797). At the end of the book Berkeley put a series of questions on analysis and geometry, one of which is 'Question 12. Whether it be possible that we should have had an idea or notion of extension prior to motion? Or whether, if a man had never perceived motion, he would ever have known or conceived one thing to be distant from another?' In these questions Berkeley in a real form underlines that connection between our knowledge of distance and extension and the sensations of movement which he pointed out in his first book, "A New Theory of Vision" (1709).

* *Loc. cit.* Introduction.

It is not an accident that the first person who with great clearness and detail worked out the question of the origin of geometrical motions from the facts which are observed in perceiving motions taking place in the visible world was UEBERWEG, one of the most remarkable German philosophers of the nineteenth century, who translated into German Berkeley's classical work, "A Treatise concerning the Principles of Human Knowledge." For Ueberweg the following four facts given us by our sensations, serve as the empirical foundation for geometry.

1. A rigid body under no constraint can move where there is no other body.

2. If one point of the body is fixed, the body though constrained in its movements can nevertheless still move.

3. If two points of the body are fixed, then although no part of it can make all the movements possible under 2, it can yet still move; in this motion a series of points which are connected together and also with the immovable points cannot move.

4. Finally, if three points are fixed, no motion of the body is possible.*

* "Die Principien der Geometrie wissenschaftlich darstellt." *Archiv. für Philologie und Pädagogik*, Band 17 (1851) p. 25.

Ueberweg analyses the consequences which flow from these experimental facts, but imperceptibly introduces some new definitions, which do not flow from his empirical facts. For example, he defines parallel lines as lines which have the same direction, and by this definition introduces Euclid's postulate. For this reason his attempt to shew that empirical facts which confirm the free movements of bodies and the possibility of rotation about a point and a straight line completely determine the nature of space, and thus serve as a foundation for Euclidean geometry, is a failure. Only the application of mathematical analysis to deduce conclusions from some fundamental postulates can guarantee that new assumptions are not introduced. Helmholtz laid the foundations for such an application of mathematical analysis.

HELMHOLTZ was led to the investigation of the problem of the facts which lay at the basis of geometry by his work on physical optics; the same problem of vision which 150 years before formed the subject of Berkeley's "Essay towards a New Theory of Vision." Already by 1855, as is evident from his excellent lecture on 'Human Vision' delivered at Königsberg on the occasion of the erection of a monument to Kant, Helmholtz was interested in this question of the origin of geometrical presentations and the relation of

the results so obtained in connection with the physiology of the organs of sensation to Kant's philosophy

Helmholtz's work on "The Facts which form the Basis of Geometry" appeared in 1868. He took as the basis of his investigations the same facts as Ueberweg, but added the new assumption that space is a numerical manifold in which 'each particular (point) is defined by the measurement of some continuous and indefinitely variable magnitudes (coordinates).'* Every movement of a point is thus accompanied by this continuous change of at least one of the coordinates.

This new assumption allows the use of mathematical analysis. The indefinite notion of distance is replaced by the existence of equations between the coordinates of two points, and mathematical analysis enables a definite form to be given to this.

4. THE PROPERTIES OF GEOMETRICAL SPACE

These investigations of the foundations of geometry, begun by Helmholtz, were finished and completed by Sophus Lie in the third volume of his "Theorie der Transformationsgruppen," published by him jointly with Professor Engel.†

* § 1, 1.

† F. Klein (*Gutachten betreffend den dritten Band der Theorie der Transformationsgruppen von S. Lie anlässlich der ersten Verteilung des Lobatschewsky Preises*) pays special attention to the relation of the work of Helmholtz and Lie to one another.

Sophus Lie applied this theory of transformations of finite continuous groups to the question of the properties of space, which he called the problem of Riemann and Helmholtz.

The possibility of doing so is founded on the fact that all displacements of a rigid body taken together constitute an infinite group. If we displace a body from position A to position B and then displace it from position B to position C , the result of such consecutive displacements will evidently be a displacement from position A to position C . This last displacement is certainly included as one of the displacements of the rigid body. It is necessary to point out that in fact no displacement from position A to position C is precisely equivalent to consecutive displacements from A to B and B to C because absolutely rigid bodies do not exist in nature. Physically rigid bodies are certainly liable to various changes according as to whether they are displaced directly from position A to position C or pass through the position B . In order to be able to consider the displacements as a group we consider each alteration of the body as equivalent to two, one of which is taken to be a displacement precisely satisfying the condition necessary that the displacement should constitute a group, the other alteration is merely one of quality. In this way the assertion that the displacements

constitute a group is the result not only of experience but also of the action of our mind.

The group of displacements is an infinite group, in which the reverse displacement from B to A corresponds to the displacement from A to B . Therefore the infinite group of displacements includes the operation of identity that is the case where the body is left at rest.

From the statement that displacements form an infinite group, possessing the property we have mentioned, important results follow. We state these by attributing to space the property of homogeneity and in particular of isotropy.

By saying that space is homogeneous we mean that all possible displacements of a body from position A are also possible for it from position B to which it can be moved from position A . But this is just what follows from the fact that the displacement constitutes a group, that is from the fact that two displacements—that from B to A (always possible so that displacement from A to B is possible) and that from A to C —may be replaced by one displacement from B to C . The same thing applies to the isotropy of space, by which we mean that all displacements possible for a body when it is in position A are also possible for it when it is in position B , to which it may be displaced from position A by rotation about a point or axis.

From the homogeneity of space it follows that it is unbounded. What has a boundary cannot be homogeneous, for boundaries have different properties to those of other places. But being unbounded is not the same as being infinite, as Ueberweg, for example, supposed. This distinction between the idea of being unbounded and that of being infinite, which was made clear by Riemann in his well-known paper, is illustrated by the surface of a sphere which is unbounded but finite.

The group of displacements is not merely an infinite group. It is also a continuous group containing infinitely small displacements. In other words, to postulate the infinite divisibility of displacements is to postulate that to each commensurable or incommensurable number there corresponds a displacement. This property of the group is expressed by the assertion that space is continuous.

The fundamental properties of geometrical space—homogeneity, isotropy, continuity, unboundedness—are deduced in the same way from the statement that displacements constitute an infinite continuous group. Ueberweg deduced the same properties from the fact that rigid bodies could move freely, which in the form of the axiom ‘things which can be superimposed on each other are equal,’ had been taken by Euclid as the basis of his geometry.

If in this way the fundamental properties of geometrical space are formulated in the works of Lie as properties of groups, then all the other properties of geometrical space (equally with the fundamental geometrical conceptions of a point and a straight line) are found to depend on the existence of *subgroups* of rotation.

The idea of subgroups has a very important place in Lie's theory, as in the theory of groups in general. If out of all the operations of a group it is possible to separate a part, which constitutes a group in itself, then this will be a *subgroup* of the complete group. Thus in relation to the group of all movements which bring an octohedron to coincide with itself, the group consisting of the four rotations (90° , 180° , 270° and 360°) round any of its axes constitutes a subgroup. So the group of transformations of the number x into the number $x + 2k$ where k is an integer, is a subgroup in relation to the transformations of x into $x + n$ where n is any number. So in relation to the group of all displacements of a rigid body, the rotations about a fixed point, the rotations about a fixed axis and the movements parallel to some line constitute separate subgroups.

The connection of the rotation of a rigid body with the foundations of geometry was noticed by Leibniz who defined a straight line as the axis of rotation or geometrical position of the motion-

less points of a body which moves when two of its points are fixed. Ueberweg, as has been pointed out above, took as the foundation of his treatise the empirical facts of free mobility and the possibility of rotation round a point and a straight line.

The theory of continuous groups made it possible for Sophus Lie to state precisely and to solve the problem which Ueberweg foresaw. Lie shewed that the group of displacements is not completely defined by the existence of subgroups of rotation (rotation about a point and about a fixed line), that there are three groups of motions which have such subgroups. These three groups correspond to the three forms of space, Euclid's, Riemann's and Lobachevski's.

5. METRICAL REPRESENTATION OF SPACE

This new fact which separates off the group of movements which corresponds to Euclidean space, and allows us in investigating space-forms to use the formulae of Euclidean geometry, is the existence of movements of translation of a rigid body, which form a subgroup of the general group of all displacements. The existence of this important subgroup is thus equivalent to Euclid's axiom of parallel lines and separates Euclid's geometry from that of Riemann and Lobachevski. In this way the investigations of

Helmholtz and Lie on the foundations of geometry as a science of measurement of lines, areas and volumes (metrical geometry) established the close connection between geometry and the theory of the groups of motion with the existence of solid bodies. This is a more accurate expression of Newton's phrase (Preface to the "Principia"): 'Geometry is based on practical mechanics and is merely that part of general mechanics which explains precisely and demonstrates the art of measurement.' POINCARÉ in his own charming laconic way has expressed the essence of the new attitude to geometry in two phrases: 'What we call geometry is nothing but the formal properties of certain groups so that we can say—space is a group.'* 'If there were no solid bodies we should not have had geometry.'†

The geometry which Poincaré refers to in the second phrase is metrical geometry (Newton's "Ars mensurandi"). The space he refers to in the first phrase is the space of the geometer, the result of a synthesis and idealisation of space presentations. But we have already had occasion to refer to the immortal service of Berkeley who first analysed our presentations of the size and distance of objects and shewed that these geometrical presentations themselves appear to be a

* See *Dernières Pensées*, p. 50.

† *La Science et l'Hypothèse*, p. 80.

synthesis (association) of our sensation of sight, touch, and motion.

The labours of thinkers strongly influenced by the English empirical school (Brown, Bain, Spencer) and of the other German philosophers (J. Müller, Helmholtz, Hering, Stumpf)—have explained the difference between the physiological space of our sensations and the space studied in metrical geometry. In all this work special attention is devoted to visual space.

Independently of these psychophysiological works, the mathematicians (Leonardo da Vinci, Desargues, Poncelet, Staudt), impelled by the demands of art and its practical applications, developed *projective* geometry, which may be called the geometry of visual space. In it we study those geometrical properties, which can be found and studied in the motionless and sensible things we can see. Using Poincaré's words, if metrical geometry studies the properties of the motion of rigid bodies, projective geometry studies the properties of light rays. Like metrical geometry, projective geometry too is closely connected with the theory of groups so that the study of its properties may be considered as that of the properties of groups, not movements but transformations. The groups of movements of rigid bodies, corresponding to the three forms of space, include projective transformations as sub-

groups and therefore, as the English mathematician Cayley shewed in 1859, the possibility of a metrical definition is connected with the separation out of the groups of projective transformations of subgroups which leave invariant a certain geometrical figure, the so-called *absolute*. Developing Cayley's idea, *F. Klein* was able not only to give a new remarkable interpretation to non-Euclidean geometry, but also in his Erlanger programme from one general point of view, to the different kinds of geometrical discipline as separate parts of the general theory of groups. This general point of view might also be expressed in the above quoted words of Poincaré: 'What we call geometry is nothing but the study of the formal properties of certain groups.'

From this point of view the so-called affine relationship, which was studied by *Möbius* in 1827 in his baricentric calculus and is very important in the mechanics of deformed bodies (hydrodynamics, theory of elasticity, etc.), can be considered as the study of the properties of *affine* (linear) groups of transformations. An affine group includes in itself those projective transformations in which a straight line at an infinite distance cannot correspond to a line in a finite part of a plane, or in other words, those transformations in which a straight line at infinity remains invariant. The analytical expressions

for these transformations in two-dimensional geometry, where the points are given by the coordinates (x, y) is

$$x' = a_{11}x + a_{12}y + a_{13}$$

$$y' = a_{21}x + a_{22}y + a_{23}$$

Extending this to manifolds of many dimensions, we obtain for example in the case of geometry of four dimensions whose points are defined by the coordinates (x, y, z, t) the analytical expression for the corresponding affine transformations in the form

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t + a_{15}$$

.....

.....

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t + a_{45}$$

i.e. the corresponding group depends on 20 parameters and from what we have said above (p. 60) may be represented by the symbol G_{20} . In the next chapter we shall see the relation of this group to the so-called *Lorentz* group, which is of fundamental importance in the special theory of relativity. Affine geometry has the same enormous importance in the general theory of relativity.

6. GEOMETRICAL AND PHYSIOLOGICAL SPACE

The investigations of Cayley and Klein in the theory of the general definition of measurement

just as the study of affine geometry shewed itself (equally with the investigations of Christoffel, Lipschitz, Levi-Civita and others in the direction given by Riemann) to be an indispensable mathematical weapon for the creation of the general theory of Relativity. No less important for the understanding of that theory was the change in the way of regarding space and the relations of geometry and physics due to the impulse given by the founders of non-Euclidean geometry. It is not possible for us to expound in detail the philosophical disputes which these new investigations led to, and we refer the reader to the works of Benno Erdmann, Russell, Couturat and others. But it is extremely important to point out that the significance of the connection, which was observed by Berkeley, Lobachevski and Ueberweg and investigated by Helmholtz and Lie, between the question of space and that of motion was clearly understood by those thinkers who took a special interest in mathematical physics. Helmholtz, Mach and Poincaré, although they had entirely different philosophic outlooks, nevertheless agreed in recognising the importance of the new points of view. Thus Helmholtz says: 'The axioms of geometry are not concerned with space-relations only, but also, at the same time, with the mechanical deportment of the solidest bodies in

motion.* At the time when Helmholtz was defending the view that the axioms of geometry were derived from experience, and yet also considered Kant's view of a transcendental space to be the true one, Mach in the first page of his most important work, "Contributions to the Analysis of Sensations," put space and time on a level with sounds, colours, and pressure. *Mach* devotes an important part of his "Erkenntnis und Irrtum," to the difference between physiological and metrical space, to space and geometry from the point of view of physics. His book contains rich material on the question of the connection between our physiological organism and spatial presentations.

Poincaré occupies an intermediate position between the empirical school and that of Kant. As Euclidean geometry is simpler and more convenient than non-Euclidean geometries, so, according to Poincaré, if the facts—*e.g.* those of astronomical observations—do not fit, science should always try to support Euclidean geometry, by varying the hypotheses of physics. It should be guided by this principle of 'convenience' which, like Mach's principle of 'economy,' enables the logic of the exact sciences to be constructed, without considering whether these foundations are absolutely true.

* "On the origin and significance of geometrical axioms." *Popular Lectures*, 2nd Series (1888), p. 65.

In this way, according to Poincaré, there is no sense in asking whether Euclidean geometry is true or what system of measurement or of co-ordinates is true. One geometry cannot be more true than another, it can only be more convenient. It is more convenient because it is simpler and agrees with the properties of rigid bodies, which we use for measurement by the organs of our body or artificial instruments. According to Poincaré, geometry is not an experimental science; the space which it studies is our creation, but it is adapted to the world in which we live. In every measurement the construction of this space is the work of a whole series of our ancestors, animals of the highest type. For each measurement space is a fact independent of the individual—that is the problem. Such is Poincaré's view on the origin of geometrical truths; it differs from the philosophy of Kant and from the empiricism of Lobachevski and the English school. Though he differs from Helmholtz and Mach, Poincaré, as we have seen, agrees with them in recognising the close connection between geometry and physics.

We are nearing the centenary of that memorable day (12 February 1826) when Lobachevski, in the Hall of the University of Kazan, read his paper on the principles of geometry. In a cursory sketch we can only point out the most important stages of the two routes taken by the work of

thought freed from the hypnotic influence of Euclid's geometry with its seemingly unshakeable axioms and its synthetic method of exposition.

On one of these routes it became clear that Euclid's geometry ought to be regarded as a particular instance of a more general mathematical discipline—the theory of manifolds of many dimensions. This general theory which borrows from geometry only the general notion of continuous magnitudes, not only lines, areas and volumes, but all those magnitudes which are studied in the various departments of natural philosophy—mechanics, physics, and psychology—was worked out by these ‘ideale Gesellen,’ to use Minkowski's phrase in his well-known address,* without any thought of its practical application.

But he continues: ‘on one fine day it turned out that a real edifice of physics might be built on the foundation of this theory.’

By the side of the workers in the direction of generalising geometry into the general theory of manifolds, there went on the other work of getting a more profound knowledge of the foundations of geometry, of our spatial presentations. This work was partly done by the psychophysicologists. Their investigations originating with Berkeley, made clear the difference between

* At the Göttingen Mathematical Society on 5 Nov. 1907. Printed in *Annalen der Physik*, Bd. 47 der 4 Folge.

geometrical space—the subject of Euclidean geometry—and physiological space, which appears to be a synthesis of various spaces obtained by the different organs of sense. They also made clear the connection of geometrical representations with the movements of rigid bodies and the properties of light rays. Of no less importance were the works of the mathematicians in which special attention was devoted to the connection between the axioms of Euclidean geometry and the properties of groups of motions and of more general transformations.

In the end this way too shewed the connection between geometry and physics. This connection is so close that, as Poincaré has shewn, to explain a new fact which seems paradoxical from the point of view of contemporary geometry and physics, we may vary the hypotheses of either physics or geometry.*

Both routes nevertheless bring geometry and physics together. Both together lead to the important position which can be summarised as follows.

The laws of Euclidean geometry are invariant in relation to groups of linear (affine) transformations, which leave unaltered the distance between two infinitely near points

$$ds^2 = dx^2 + dy^2 + dz^2.$$

* See *La Science et l'Hypothèse*.

The aim of the two following chapters will be to shew how the new results obtained in electrodynamics and their relation to the fundamental problems of mathematical physics led to the final fusion of geometry and physics and extended to the laws of space-time (a four-dimensional manifold whose elements are defined by four numbers, x_1, x_2, x_3, x_4) the idea of invariance in relation to the group of affine transformations which leave unaltered the function

$$\begin{aligned} ds^2 &= g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 \\ &\quad + 2g_{12}dx_1dx_2 + 2g_{13}dx_1dx_3 + \dots \\ &= \Sigma g_{ik}dx_id x_k \quad (i, k = 1, 2, 3, 4). \end{aligned}$$

The fusion of geometry and physics appears in the fact that the ten coefficients g_{ik} determine metrical relations (lengths and times) and also the properties of the gravitational field which gives Einstein the right to call them gravitational potentials.

CHAPTER IV

THE SPECIAL THEORY OF RELATIVITY

Siehst du, mein Sohn,
Zum Raum hier wird die Zeit.

R. WAGNER, *Parsifal*.

I. THE ELECTRO-MAGNETIC THEORY OF LIGHT

A new source of our knowledge about Nature arose almost unexpectedly and changed our physical view of the world to an extent which is not yet ascertained.

This new knowledge is closely connected with the evolution of the ideas of space and time. When, in 1820, Oersted, an able man but a bad experimenter, after persevering thirteen years, at last succeeded in shewing the influence of an electric current on a magnetic needle, no one could foresee that his discovery and the study of electro-magnetism would have to-day produced the practical results of increasing the power of man over Nature, and bringing the peoples of the earth nearer together. The study of electro-magnetism has had an equally stupendous influence on the fundamental problems of physics.

In the middle of the nineteenth century, when the continental physicists (W. Weber, F. Neumann, Helmholtz) were trying to explain the new phenomena, they, under the influence of the

immense successes of celestial mechanics, took as their basis the dualism of space and matter, and expected to be able to explain the phenomena in terms of action at a distance. FARADAY (1791–1867) introduced the new idea of a '*field*.' Instead of considering the primary cause of all electrical phenomena to be electrical charges, endowed with a power of acting at a distance, Faraday considered these phenomena to be due to a state of strain of the electric field, which appears as 'lines of force.' MAXWELL (1831–1879) invested Faraday's ideas with a mathematical form of unusual simplicity and beauty, and formulated those equations to which Boltzmann justly applied the words 'War es ein Gott, der diese Zeichen schrieb?''*

In the Faraday-Maxwell theory the electromagnetic field or ether is the medium of electromagnetic phenomena; the state of the field at each point is characterised and completely determined by two vectors—the electric and the magnetic—whose variations in space and time are connected by the differential equations discovered by Maxwell. Coulomb's law, that the force is inversely proportional to the square of the distance and does not involve time, is a law of action at a distance—an action instantaneously transmitted without a medium. If we replace

* *Faust*, Part I. l. 434.

action at a distance by a continuous medium, requiring time for transmission, we must use differential equations; that is, equations which connect infinitely small spatial changes of the strain of electrical and magnetic force with infinitely small changes in space and time (*i.e.* space and time coordinates reckoned from any moment). Maxwell's field equations and the electro-magnetic theory of light based on these were verified experimentally by Hertz's (1888) discovery of electro-magnetic waves and by P. H. Lebedev's (1890) discovery of the pressure of light (confirmed by Nicholson and Hull in 1904).

After these discoveries the ether of Huygens' wave theory of light (which had been defended by Euler in the eighteenth century and revived at the beginning of the nineteenth by Fresnel and Young) attained an added importance as the medium not merely of light phenomena but also of electro-magnetic ones.

2. THE LORENTZ HYPOTHESIS

At the same time the problem of the relation of the ether to the motion of matter became of increasing importance. This question already had a long history. Fresnel had already dealt with it, and before the discovery of the electro-magnetic theory of light three possible theories

of the relation of the ether to a moving body had been propounded:

1. The theory that the ether is completely dragged along by the body;
2. The opposite theory that while the body moves the ether remains completely at rest;
3. The theory that the ether is partly dragged along.

The first two directly contradict each other. While according to Stokes's theory the ether inside the body entirely partakes of its motion; according to the second theory, on the other hand, material bodies move through the ether without dragging it with them, just as a fishing net goes through the water of a river. The hypothesis that the ether was partly dragged along is due to Fresnel and was considered to be fully confirmed by Fizeau's experiment with flowing water and to be the only explanation of this experiment. Fresnel's hypothesis explained not only this experiment, but also all the other experiments, in which optical phenomena depend only on the relative motion of material bodies *inter se*, and are independent of their motion relative to the ether (optical principle of relativity). But the phenomena are explained if we assume that the ratio β between the velocity of the body v and the velocity of light c ($\beta = \frac{v}{c}$)

is so small that its square may be neglected. In the case of the earth's motion ($v = 30$ kil. per sec.) $\beta = \frac{1}{10^4}$. Before 1881 the technique of optical experiments did not make it possible to investigate phenomena which depended on the square of β (second order facts). From the time of the discovery of the electro-magnetic theory of light, both the structure of the ether and the electro-dynamics of moving bodies became the most important problems of physics, and various solutions were given. At the time when the English physicists (Maxwell himself, and especially Lord Kelvin) were trying to make models of the structure of the ether, and to explain electro-magnetic phenomena by means of them, Hertz deliberately rejected all mechanical explanations of these phenomena and produced a strictly phenomenological theory. 'We do not know,' he writes, 'what is the essence of those changes of physical state which are called electrical and magnetic disturbances. All we know is the phenomena caused by them.' For the fundamental problem of the electro-dynamics of moving bodies, he adopted the hypothesis that the ether was completely dragged along. But though all the results of his theory were completely confirmed by experiments with moving conductors (Roland, Eichenwald) yet experi-

ments with moving insulators (Röntgen and Eichenwald in the electrical field, and Wilson in the magnetic field) gave results which contradicted Hertz's theory.

In opposition to the phenomenological theory of Hertz, LORENTZ constructed a theory of electricity based on the hypothesis of an atomic structure of electricity; this was first suggested by Helmholtz (1881) and had been confirmed by the discovery of the cathode rays. According to this theory all electro-magnetic phenomena consist in the movements of electrons and the fields produced by them. Similarly as regards the question of the movement of bodies in the ether and the properties of ether, Lorentz, in opposition to Hertz, developed the hypothesis of an ether which was motionless in space.

The mathematical theory of Lorentz was able to explain all optical phenomena by neglecting quantities of the second order, *i.e.* quantities of the order of smallness of β^2 ; but at the same time this theory shewed that it would be possible to determine the motion of the earth relative to the ether (*i.e.* absolute motion) by means of optical experiments if a method could be found of measuring quantities of the order of β^2 . But even before Lorentz had formulated his theory, an American physicist, *Michelson*, had constructed an extremely delicate interferometer,

which would be able to detect the 'ether wind.' But the experiments he made in 1881 and again in 1887 (in conjunction with Morley) and the more recent experiments of Morley and Miller (1905) gave a result which contradicted the hypothesis of a motionless ether.

'There is no ether wind. The velocity of light is independent of the motion of the earth through the ether, even if we take into consideration quantities of the second order.' Michelson himself drew the conclusion from his experiment that the ether was completely dragged along by the earth (as in the theories of Stokes and Hertz), but this conclusion is contradicted by numerous experiments. Ritz* (1908) thought it possible to explain Michelson's experiment by starting from the assumption that the velocity of light depends on the velocity of the source of the light. But this hypothesis, too, is contradicted by numerous experiments (*e.g.* Arago's); and recently (1913) the Dutch astronomer, de Sitter, proved by careful investigations that the velocity of light coming from the fixed stars, is independent of the motion of these stars. A daring hypothesis of another kind was propounded first by an Irish scientist, *FitzGerald* (1892), and afterwards developed in detail by

* See Pauli, "Relativitätstheorie." *Encyclopaedie der Mathematischen Wissenschaften*, Band II. 1921.

Lorentz. According to this hypothesis, which completely explains Michelson's experiment, every body moving with a velocity v in relation to the ether is shortened in the direction of the motion by a fraction, which in the case of the movement of the earth is so minute that the diameter of the earth is only diminished by six centimetres. This contraction is very small; but even if it were considerable, yet it could not be discovered because every terrestrial measuring-rod would be shortened in just the same ratio; only an observer who was at rest in the ether could observe this contraction of the earth and all objects on it. The theory of Lorentz explains not only Michelson's experiment, but also many others which, like it, shewed that we cannot discover the motion of the earth in the ether even if we take into consideration quantities of the second order.

The most important of these was that of Trouton and Noble (1903) designed to determine the force of rotation which should have arisen as a consequence of the influence of ether wind on a suspended flat condenser. All these experiments led to the conclusion that the uniform motion of translation of the earth through the ether could not be discovered by an observer on the earth who took part in the earth's movement. In other words, *the principle of relativity which*

applies to all mechanical phenomena also applies to all electro-magnetic phenomena.

Lorentz (1899) found that in order to deduce mathematically from his electron theory with a motionless ether the fact that all electro-magnetic phenomena take place in moving systems exactly as they do in stationary ones, it was not sufficient to adopt the hypothesis of contraction, but that it was necessary to adopt an equally strange one, viz. that in different moving systems there are different measures of time; consequently every uniformly moving system has its own proper *local* time.

The two hypotheses taken together assert that in moving systems both space and time are measured in a different way to that in stationary ether, and both hypotheses appear at once to be a consequence of one and the same remarkable mathematical fact, a property of the equations of the electron theory, and also of the slightly different field equations of Maxwell. This was first noticed by VOIGHT (1887) in regard to the equation for the propagation of light deduced from Maxwell's equations. This remarkable fact is that Maxwell's equations, like the equations of classical mechanics, are invariant for a certain group of transformations. In this way, although the hypothesis of a stationary ether was the starting-point of Lorentz's work, his mathe-

matical theory gave rise to an entirely new way of regarding the problem of electro-magnetic phenomena in moving systems. Instead of solving the problem whether the ether is stationary or not, the theory gave a new form to the principle of Relativity different from that of classical mechanics. Just as the invariance of the mechanical equations in relation to group N (cf. chap. II) shewed the impossibility of determining position in absolute space by mechanical means, so the invariance of Maxwell's equation in relation to a new group shewed the impossibility of determining position in the absolute ether. But this deprives the ether of its character of a substance, a medium possessed of physical qualities; and so the theory of Lorentz leads to a new view of the ether as only an auxiliary means of studying electro-magnetic phenomena; that is, a field, *i.e.* a collection of points each of which has an electro-magnetic vector corresponding to it. But none the less, neither Lorentz nor Poincaré in his paper "Sur la dynamique de l'électron" (1905), which is a very remarkable piece of mathematics, departed from the usual point of view. They considered the FitzGerald contraction to be an actual physical phenomenon which they explained by means of a hypothesis about the laws of action of electro-magnetic forces. A different and much more profound way of re-

garding both the new form of the principle of Relativity and the consequences which flowed from it (the FitzGerald contraction and the local time of Lorentz) is due to EINSTEIN.

3. THE MATHEMATICAL EXPRESSION OF THE PRINCIPLE OF RELATIVITY

But before coming to Einstein's solution, which has a profound epistemological importance for the theory of space and time, we must pause at the new form of the principle of Relativity, *i.e.* at that group of transformations which expresses the principle analytically. The form of the new group is essentially different from that of the group N , which leaves the equations of classical mechanics invariant, although both groups are groups of affine (or linear) transformations. We will denote the new group by the letter L .

In the case of group N , the distance between two points of a moving system (*i.e.* taking one of the points at the origin so that its coordinates are zero), the function $x^2 + y^2 + z^2$ is invariant. The time t does not enter into the invariant of the group. The reason for this is found in the fourth equation of group N $t' = t$, *i.e.* in the assumption that for all moving systems there is absolute identical time. On the other hand, the invariant of group L , since it comes from the invariants of the equation of the propagation of

light (which Voight had found) will be the expression $x^2 + y^2 + z^2 - c^2 t^2$, where c is the velocity of light derived from Maxwell's equations.

The general equations of group L ought, thus, to be obtained if out of the general groups of affine transformations of the variables x, y, z, t we select that subgroup which has for an invariant $x^2 + y^2 + z^2 - c^2 t^2$, or (a little more generally) the equation $x^2 + y^2 + z^2 - c^2 t^2 = 0$.

The difference between the invariants of the two groups is what distinguishes the new form of the principle of Relativity from that of classical mechanics. The essence of the special theory of Relativity due to EINSTEIN, lies in the explanation of this mathematical fact.

The difference between the two groups of transformations follows from the differences between the invariants. To avoid as far as possible complicated mathematical calculations we will confine ourselves to the particular case where $y_1 = y, z_1 = z$. The problem then consists in selecting from the general group of linear transformations

$$x_1 = mx + nt,$$

$$t_1 = px + qt,$$

those transformations in which the equation $x_1^2 - c^2 t_1^2 = 0$ transforms into the equation

$$x^2 - c^2 t^2 = 0.$$

We have to solve three equations with the four unknowns m, n, p, q .

The solution is considerably simplified if we put $n = -mv$. We then obtain the result

$$x_1 = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y_1 = y, \quad z_1 = z, \quad t_1 = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Solving for x, y, z, t , we obtain

$$x = \frac{x_1 + vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y_1, \quad z = z_1, \quad t = \frac{t_1 + \frac{v}{c^2}x_1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Group L is a generalisation of group N .

It is easily seen that if in the first we make c infinite or, in other words, v infinitesimal in relation to the velocity of light, we obtain the second group in which v has the physical meaning of the velocity of the uniform motion of the system x_1, y_1, z_1 relatively to the stationary system x, y, z . This is why in general Lorentz and Poincaré had the right to consider the equations of group L as equations which determined the movement of a system in which the position of a point is defined by the coordinates x_1, y_1, z_1 and the time by t_1 in relation to another system defined by x, y, z, t . The motion will likewise

be one of translation in the direction of the x axis with a constant velocity v . The invariance of Maxwell's equations in relation to group L explains why no electro-magnetic or optical experiments can possibly 'determine the absolute motion of the universe' (Poincaré), *i.e.* determine which of the two systems is the one in motion and which is the one at rest.

In this way the difference in the structures of the groups N and L represents in itself, both for Lorentz and for Poincaré, a mathematical fact.

The paradoxical result follows that a rod moving with a velocity v in the direction of the x axis experiences a contraction in the ratio of

$$\sqrt{1 - \frac{v^2}{c^2}} : 1;$$

a sphere becomes an ellipsoid of revolution. Both these were explained by a new hypothesis as to the law of action of electro-magnetic forces.

However, at the end of the preface to his memoir on the dynamics of electrons, Poincaré makes a very interesting observation of principle: 'The part which is, so to say, common to all physical phenomena will only be an appearance, something due to our methods of measurement. How do we measure? By superposing one on another objects which are regarded as rigid and invariable; but if one admits the Lorentz con-

traction that is no longer true. In this theory by definition two lengths are equal when they are traversed by light in equal times.

To give up this definition would perhaps suffice to overthrow the theory of Lorentz, as completely as the Ptolemaic system was overthrown by Copernicus. Should this happen, it will not prove that the effort made by Lorentz has been useless. Ptolemy, whatever one thinks of him, was useful to Copernicus.*

4. THE RELATIVITY OF SPACE AND TIME

The new statement of the problems connected with the Lorentz group, which Poincaré foresaw in words just quoted, was made in the same year (1905) by Einstein in his celebrated paper: 'On the Electrodynamics of moving bodies' (*Annalen der Physik*, Bd. 17).†

In his opening sentences Einstein traces the difficulties and the inconsistencies of the electrodynamics of moving bodies with the general principles of physics: 'The theory to be developed is based—like all electrodynamics—on the kinematics of the rigid body, since the assertions of any such theory have to do with the relationships between rigid bodies (systems of

* "Sur la dynamique de l'électron." *Rendiconti del circolo matematico di Palermo* (1906), vol. 21, p. 3.

† *The Principle of Relativity*, by A. Einstein and others, contains this and other papers translated into English.

coordinates) clocks and electro-magnetic processes. Insufficient consideration of this circumstance lies at the root of the difficulties which the electrodynamics of moving bodies at present encounter.*

We cannot avoid investigating the connection between phenomena and the apparatus used for measuring space and time. But what kind of space, what kind of time? Einstein at this first stage of development of his critical ideas confines himself to the question of time.

The French philosopher, Le Roy, draws a clever comparison between the way in which time is regarded by the philosopher and the physicist. 'Ask the former what is time and he answers you in a long speech, the latter takes out his watch and says "Here it is".'

A good illustration of this remark may be found by comparing the nebulous pages of Aristotle's "Physics" with Lobachevski's definition of time in his lectures on mechanics (preserved in the Bibliotheca Lobachevskaya): 'The motion of one body, if it is taken as the measure of the motion of another body is called time.'

As a matter of fact, a definition of time always comes to be a definition of the simultaneity of two events. It comes to the same thing whether

* *The Principle of Relativity*, by A. Einstein and others, p. 38.

I say that I woke up at sunrise or when the hand of my watch pointed to seven o'clock, or finally when a noise in the adjoining room aroused me. But what is this simultaneity?

His criticism of the idea of simultaneity or, in other words, of absolute time, constitutes the essential basis of Einstein's statement of the new form of the principle of Relativity. It is doubtful whether anyone to-day denies the importance of the mathematical fact that Maxwell's equations are invariant for transformations of group L , which lead to this new form. All electrodynamics and its practical applications are based on Maxwell's equations just as all classical mechanics with its applications to science and life are based on the equations of motion of a point given in chapter II (p. 54). But the interpretation of this mathematical fact still gives rise to differences of opinion.*

* So Lorentz, whose name is indissolubly connected with the new form of the principle of Relativity in his Haarlem lectures, delivered in 1917, says: 'It is in the main the business of the theory of knowledge to estimate the value of the fundamental ideas of the theory of Relativity; we must therefore await its judgment hoping that it will consider all the difficult problems of relativity with the necessary thoroughness. But it will certainly depend to a considerable degree on the type of thought to which one is accustomed, whether one feels himself drawn to one or other view. I personally still find a certain satisfaction in older views that the ether still preserves a certain substantiality, that space and time are sharply

Einstein's fundamental position may be stated as follows: Two systems are moving uniformly in a straight line relatively to each other; on one is an observer *A*, on the other *B*: each observer considers his system to be at rest. *Two events which appear simultaneous to one observer will appear non-simultaneous to the other.* Such an assertion is in obvious contradiction to the usual view held since the time of Newton's "Principia" that Time 'flows uniformly without any relation to any outside object'; and we who are wont to think that we understand the truth and unshakeableness of the Newtonian definition find the statement difficult to grasp. But contemporary physics are becoming more and more accustomed to take as their basis the principle which Planck formulated in the words: 'What can be measured is what exists.' We can only attribute physical reality to something which is not merely observable but measurable. From the point of view of this principle it is difficult to find an objection to Einstein's assertion. The idea of two events being simultaneous, which is distinguished, and that one can speak of simultaneity without a closer specialisation.' (Published in *Zeitschrift für mathematischen und Naturwissenschaftlichen Unterricht*, 1917.) A. N. Whitehead, in his profound book on the *Principle of Relativity*, and Bergson, in his *Durée et Simultanéité*, maintain the old belief in the fundamental character of simultaneity. [Author's note.]

clear when both happen at the same place, ceases to be clear when we are speaking of events happening at different places. To define it we need the use of sound or light signals or other physical apparatus, such as a chronometer.

In fact, if two events occur at the same place the conviction that they are simultaneous is equivalent to the conviction that they both occur when the hand of a watch, which is near them, is at a particular position on the dial. But if two events, one at Kazan and the other at Vienna, occur simultaneously, it requires science and scientific apparatus to establish their simultaneity. There is a tradition in my family that the portrait of the astronomer Littrov, my grandfather's* tutor, fell down in the astronomical cabinet of Kazan University at the precise moment when Littrov died at Vienna. Were these events actually simultaneous? Even in the case of events on the earth it requires a knowledge of the difference of the times of Vienna and Kazan to establish simultaneity, that is, a knowledge of their longitude, and further, that the clocks which determine the time at two different places had been regulated or, as we say, synchronised (have an identical rate). It is a very much more

* Professor Simonov who, as an astronomer, took part in the Bellinghausen expedition, 1821. One of the islands of the Fiji group is named after him. [Author's note.]

difficult task when one of the two events is on the earth and the other on the sun or Arcturus. To solve the problem we must first of all know the velocity of light. The man in the street thinks that everything happens at the moment it is presented to our consciousness. In Eddington's clever phrase—time as we understand it now, the time of the universe, was discovered by Roemer. The discovery that light had a finite velocity compelled us to give up the old illusion that the moments of our consciousness have a universal significance, and are related to all the visible universe. We know to-day that two phenomena which we observe in the vault of heaven—the outburst of a protuberance on the sun, or the appearance of a new star—occurred in the first case more than eight minutes before we observed it and in the other when not merely we, but human beings did not exist on the earth. The discovery of the finite velocity of light had for celestial phenomena the same immense importance as the determination of the velocity of sound had for events occurring on the earth. The knowledge of the velocity of sound enables us (on the hypothesis that light has an infinite velocity, *i.e.* arrives simultaneously) to determine the distance of a flash of lightning or the discharge of a shell from an enemy gun. Knowing the velocity of light we are able, by calculating

the distance of a fixed star from its parallax, to measure the duration of time which elapses between a celestial phenomenon and the moment when it is perceived by our eyes. But to determine the distance between two systems the knowledge of the velocity of light alone is insufficient if one or both systems are in motion. To understand this important assertion it is useful to apply the analogy between sound and light signals.

Let us consider a system consisting of a steam tug and a barge attached to it by a taut rope. Suppose that at 12 o'clock the steamer gives a sound signal to the barge. If the steamer and the barge are at rest and if we know the velocity of sound and the length of the rope we can regulate the clock on the boat so that the time indicated at each moment by this clock coincides with that of the clock on the steamer; assuming, of course, that both clocks go at an ideally accurate rate.

To do this it is necessary when the sound signal reaches the barge to set the hand of the clock not at twelve but to advance it by a seconds; $a = 1$ if the rope is 340 metres long (the distance traversed by sound in one second). But if the steamer and the barge are in motion, it is necessary to remember that the barge is going to meet the sound waves of the signal made on

the steamer and therefore in order to regulate accurately the clock on the barge with the clock on the steamer the hand of the clock on the barge must be advanced not a seconds but b seconds, where b is less than a . Evidently the difference between a and b depends on the velocity of motion of the system. Now suppose that there are two systems, each consisting of a steamer and a barge, that one system is at rest and the other in motion. Two events which are simultaneous for an observer on the first system will not be simultaneous for an observer on the second.

In our case of systems on the earth using sound signals both systems may avoid the difference in the times of their clocks by having recourse to light signals. But in the case of celestial bodies moving in space with light signals instead of sound signals, the difference cannot be avoided and we see that we have to abandon the idea of absolute time.

If there are two systems moving with velocity v relatively to one another, which have the coordinates x, y, z for the one and x_1, y_1, z_1 for the other, and the time coordinates t and t_1 , is it possible to find a rule for passing from time t to time t_1 ?

Einstein holds that this problem can be solved if we start from the postulate that the velocity

of light is constant and independent of both the direction and the velocity of motion of the system. The mathematical statement of this postulate is contained in the assertion that the velocity of light for the one system which is

$$\frac{\sqrt{x^2 + y^2 + z^2}}{t},$$

and that for the other which is

$$\frac{\sqrt{x_1^2 + y_1^2 + z_1^2}}{t_1}$$

is the same constant c ; *i.e.* that the equation

$$x_1^2 + y_1^2 + z_1^2 - c^2 t_1^2 = 0$$

can be deduced from the equation

$$x^2 + y^2 + z^2 - c^2 t^2 = 0.$$

We start then from the assumption that x , y , z , t and x_1 , y_1 , z_1 , t_1 are connected by single valued linear transformations* to the general group of Lorentz transformations depending on the 20 parameters which we spoke of in the preceding chapter and which may be represented by the symbol G_{20} . Group L , which we have given above, is a particular case of the transformation when $y = y_1$ and $z = z_1$.

* This character of the transformation comes from the requirement that motion which is uniform in relation to one system should be uniform in relation to the other. [Author's note.]

Using the formulae of this group we obtain the following results for the measurement of the length of a rod and the interval of time between two events.

1. If the length of a rod measured in system S in which it is at rest is l , the same length measured in system S_1 moving with a velocity v relatively to system S will be

$$l' = l \sqrt{1 - \frac{v^2}{c^2}},$$

i.e. the rod will appear to be shortened. But conversely, since the system S may be considered as moving relatively to S_1 , the length of a rod at rest in system S_1 when measured in system S will not appear lengthened, but similarly shortened.

2. The same thing takes place in relation to the measurement of intervals of time between two events. The clocks of every system which is moving relatively to another will seem to go slower in that system which considers itself to be at rest. The time indicated by clocks in the system in which they are at rest is called 'the proper time' of the system. Under the name of 'local time' it has already been introduced by Lorentz, but in his theory it only appeared as an auxiliary mathematical quantity as opposed to true absolute time. Einstein's theory is that

there is no possible way of determining this absolute time, of selecting it from the infinite number of equivalent local times of different moving systems. We thus have to admit that absolute time has no physical reality; all statements of time are only significant in relation to a definite system.

The relativity of length is indissolubly connected with the relativity of intervals of time. The same rod, the same interval of time are expressed by different figures if they are measured in different systems. Each length, each interval, is actual for the system in which it is measured.

All these statements appear paradoxical to us who are accustomed to the idea of absolute time and absolute length. And an even more paradoxical consequence follows from these fundamental assertions. Every event in one moving system must appear to be happening more slowly if we measure it from another which the observer takes to be at rest. This applies both to the perturbations of atoms and to life processes. Langevin* has drawn a fantastic picture of a ball hurled from the earth with a velocity approaching that of light. Time goes slowly in the ball, life processes, too, go slower and when after two years reckoned by its chronometer the

* "L'Evolution de l'Espace et du Temps." *Scientia*, x. (1911), at p. 50.

ball returns to the earth, two or three hundred years (measured by clocks on the earth) will have elapsed. The traveller, who has scarcely altered at all, will see a new earth.

5. MASS AS A FORM IN WHICH ENERGY APPEARS

We will now turn to some other consequences of the special theory of Relativity. In the first place we will pause to ask what is its relation to classical mechanics whose theories and formulae have been justified by the striking discoveries based on them (both in the science of the microcosm and in that of the macrocosm) on the one hand, and, on the other hand, by their many and varied applications to life. But we have already shewn that the formulae of the special theory of Relativity become those of classical mechanics if we put $c = \infty$. Hence for all velocities which are negligible in comparison with that of light (such as the velocities of the heavenly bodies and of the molecules in a gas), the formulae of the special theory of Relativity lead to results which do not differ perceptibly from those of classical mechanics.

The only kind of experiments which can be decisive, are concerned with those phenomena in which we meet with velocities approaching that of light. We shall shew further that in that part of physics which studies such phenomena

the formulae of the special theory of Relativity have been verified. As a preliminary it was necessary to construct mechanics and electrodynamics on the basis of the formulae of the special theory of Relativity. This was done partly by Einstein himself in his paper of 1905,* partly by Poincaré's work referred to above. It was completed with great mathematical elegance by Minkowski. As a result, those optical problems (aberration, the Döppler principle, Fresnel's formula) which had for a long time presented the greatest difficulty were solved more accurately and clearly.

We cannot go into the detail of the solution of these problems and of the new mechanics of Einstein and Minkowski; we will only pause on the kind of change which the new mechanics has made in the fundamental formula of kinematics—that for the composition of velocities.

The deduction of this formula from those of group L (which is not difficult for the reader who is acquainted with the differential calculus), gives for the velocity w compounded of the two velocities v and v' the expression

$$w = \frac{v + v'}{1 + \frac{vv'}{c^2}}.$$

* *The Principle of Relativity*, by A. Einstein and others, p. 37.

A remarkable result follows from this. If we put $v' = c$ we obtain

$$w = \frac{v + c}{1 + \frac{vc}{c^2}} = c,$$

thus the velocity of light appears to be a limiting velocity. From the formula for the composition of velocities we can deduce that for the composition of accelerations; and so can write down the fundamental formulae of Relativity dynamics. In the case of the motion of a particle whose mass at rest is m_0 we obtain an equation which is identical in form with the equation of motion of a particle in classical mechanics, but the expression for the mass which enters into it depends on the velocity of the motion and is expressed in terms of the 'rest' mass m_0 by the formula

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

This relation between the mass of a body at rest and in motion is remarkable and new consequences of great importance for physics follow from it. The formula had already been deduced by Lorentz in 1904 and applied by him to explain observations on the velocity of cathode and β rays, emitted by radio-active substances.

These observations shewed that the diminution of the ratio $\frac{e}{m}$ (e is the electrical charge on an electron and m is the mass of an electron) which varied with the velocity of the motion was particularly noticeable for β rays whose velocity approximates to that of light. This diminution might be explained as due to the diminution of the charge or to the increase of the mass. The former is inconsistent with the basis of the electron theory. The latter, on the contrary, coincides with a remarkable consequence deduced by J. J. Thomson from Maxwell's theory. The celebrated English physicist had already shewn in 1880 that the inertia of a body ought to increase if it received an electrical charge. The force requisite to set in motion a charged body at rest is greater than that required for an uncharged body. From this the conclusion might be drawn that at any rate a part of the mass was of electro-magnetic origin; and it was a probable hypothesis that an electron had in general no ordinary mass, but that all its mass was of electro-magnetic origin.

Two hypotheses were put forward to explain quantitatively the phenomena that $\frac{e}{m}$ diminished with the velocity. One of them (Abraham's, 1903) was based on the formulae of classical

mechanics and the supposition that an electron is a rigid body in the form of a sphere. On the other hand, the hypothesis of Lorentz (1904) was based on an adaptation of the FitzGerald hypothesis and the supposition that an electron in motion changed its spherical form to that of an oblate spheroid. Experiments recently undertaken by Bucherer, Hupka, Ratnovski and others agree better with the formula of Lorentz than with that of Abraham, and may be considered as serious arguments in favour of Relativity mechanics, one of whose results is the Lorentz formula. But a more remarkable confirmation of the Lorentz formula is obtained in a field which apparently has little connection with mechanics; namely, in the spectroscopy of light and Röntgen rays. The emission of light from atoms appears to be the result of oscillatory motions performed by the electrons bound within the atoms. These oscillations set up electromagnetic waves. In 1913 a Danish scientist Niels Bohr, by applying Planck's quantum theory (1900) to explain spectra, gave a remarkable model of the structure of atoms. In Bohr's hypothesis the electrons move in circles round the nucleus, and his formulae were based on the principles of classical mechanics. Sommerfeld* (1915), assuming that the electrons move in

* *Atomic Structure and Spectral Lines.*

ellipses and at the same time applying the formula of the special theory of Relativity, shewed that each line of the spectrum ought to consist of a system of more or less clear lines. The 'fine structure' of the lines of hydrogen and helium calculated by him was confirmed by the observations of the most competent spectroscopist, Paschen (1916). In 1917 Glitscher, a pupil of Sommerfeld's, found that by observing the spectrum of helium it was possible to answer the question which of the two formulae for the mass (Abraham and Lorentz) agreed best with observation and shewed that the formula of Lorentz was confirmed. Thus at the present time we can say that the special theory of Relativity has been strikingly confirmed by spectroscopy. We have already shewn how Lorentz on the basis of the electron theory with a motionless ether was led to conclude from the formula connecting mass with velocity that mass is of electro-magnetic origin. From the same formula Einstein deduced a result of the special theory which leads to a more profound view of the nature of *inertial* mass. This is that if a body receives an increase of energy E then its inertial mass is increased by $\frac{E}{c^2}$, where, as before, c represents the velocity of light. This consequence of the special theory of Relativity contradicts one

of the fundamental assumptions of classical mechanics—that the mass of a body is a constant quantity (a quantity of matter according to Newton's definition). Mechanics takes this mass as a datum which requires no explanation. The special theory of Relativity leads to the result that the inertial mass of a body is a measure of the store of energy contained in it and is equal to $\frac{E}{c^2}$.

This result is undoubtedly of immense importance. In the first place, it follows that the law of the conservation of mass is identical with the law of the conservation of energy, and that mass may be considered as a form in which energy appears.

On the other hand, if we remember the enormous velocity of light, the formula $m = \frac{E}{c^2}$ shews that a small quantity of matter must contain an enormous store of energy; in fact, a gramme of water contains 9×10^{20} ergs. Since the work of lifting a ton (1000 kilogrammes) to the height of one kilometre is equal to 10^{14} ergs, in one gramme of water, we have a sufficient store of energy to raise a load of one million tons to the top of the highest mountain (9 kilometres). Einstein calls the universal relation between mass and energy the law of the inertia of energy,

and sees in this law a most important result of the theory of Relativity. In fact, the identification which it makes between the two fundamental ideas of mass and energy ought undoubtedly to throw light on the structure of matter, that is, on the problem how material atoms are built up. From this law it follows that there is a loss of energy when an atom breaks up; the sum of the masses of the fragments should be less than the mass of the original atom. This explains neatly why the accurate atomic weights of elements do not agree with Prout's hypothesis (1815), namely that the atoms of all the elements are composed of atoms of hydrogen so that their atomic weights ought to be multiples of the atomic weight of hydrogen. Accurate measurements of atomic weights did not support this consequence of Prout's hypothesis, and for a time upset it. But the divergences of the atomic weights from whole numbers (especially at the beginning of the table of the Periodic System) are so small that this can scarcely be due to mere chance. Thus, according to Strutt, the probability that the smallness of the divergence of the atomic weights of nine elements (Br, C, Cl, H, N, O, K, Na, S) from whole numbers is due to chance is not greater than one-thousandth. To-day, Einstein's law and the work of Rutherford and Aston have explained the discrepancy, and

Prout's hypothesis that hydrogen is the primary element, has again become important in that new science which is being built up before our eyes, and which we may be allowed to call meta-chemistry. A branch of pure mathematics which formerly had little application to natural science—the theory of integers—will find evidently an extended field for its application (Aston's law of whole numbers).

6. THE WORLD AS A FOUR-DIMENSIONAL MANIFOLD

The results we have arrived at are so important for all branches of physics that it is necessary to become reconciled to the fact that many of the deductions from the formulae of the theory of Relativity are paradoxical. But if we bear in mind the important part which seeming paradoxes have played in the history of humanity, it is necessary to ask whether these paradoxes are merely seeming ones arising from our limited outlook on the world conditioned by the limitations of our psycho-physiological organisation. Undoubtedly this organisation is a factor of the highest importance in the way the world is presented to us. The question, if put in that way, certainly gives rise to most important problems in the theory of knowledge and philosophy in general. We intend to deal with them at the end of this book. Meanwhile,

we will once again call attention to the fact that the peculiarity of Einstein's views consists in his criticism of the idea of absolute time, which he replaces by relative time, depending on the state of motion of the system, and in the idea of the indissoluble union of space and time. But there is much to be said not against but in favour of the close connection between space, time and motion. Special attention was paid to this by the able mathematician, Herman MINKOWSKI* (1864-1909), who after some remarkable work on the subject of whole numbers began to apply his mathematical ability to the new subject of mathematical physics discovered by Lorentz and Einstein. Already in 1907 he delivered at Göttingen an interesting lecture which we have quoted in the preceding chapter. In 1908 he delivered an address on 'Space and Time.' In this remarkable address he insisted on the fact that the connection between the space and time coordinates given by the formulae of the Lorentz group is not accidental, but exhibits that inner connection between space and time to which sufficient attention had not been paid.

* In *Russia's Gift to the World*, produced by English scientists at the beginning of the war, Minkowski, with Lobachevski, is called a great Russian mathematician. This is an error. Minkowski, born in the Government of Kovno, was in his childhood taken by his parents to Germany to be out of the way of pogroms. [Author's note.]

‘The objects of our perception invariably include places and times in combination. Nobody has ever noticed a place except at a time, or a time except at a place.’*

To-day the dogma of the independence of space and time is shaken by new points of view arising not as the result of metaphysical reflection but from the soil of experimental physics. ‘Therein lies their strength. They are radical. Henceforth, space by itself and time by itself are doomed to fade away into mere shadows and only a kind of union of the two will preserve an independent reality.’† In order to emphasise the connection Minkowski introduces the general idea of the *world* in the place of separate space and time independent of one another. Every event takes place at a point of space defined by three numbers (space coordinates) x, y, z , and at a moment of time defined by a fourth number (time coordinate) t . The system of these numbers defines an *element* of a four-dimensional manifold. The aggregate of all these elements, a four-dimensional manifold, is the *world* or space-time. We shall sometimes make use of the expression ‘space-time continuum’ to emphasise the fact that the numbers x, y, z, t vary continuously,

* *The Principle of Relativity*, by A. Einstein and others, p. 76.

† *Ibid.* p. 75.

taking all real values from $-\infty$ to $+\infty$. An element of the world is a world point. In the world there is something, as Minkowski says, which we must not call 'matter or electricity,' as these terms are used as the basis of a physical hypothesis; it is more correct to call it *substance*. Every *substantial* point is at a given moment of time t at a definite point of space (x, y, z) , that is, coincides with the world point (x, y, z, t) . To a continuous variation of the time t there corresponds a continuous variation of the three other coordinates of the world point, that is, a substantial point describes in a four-dimensional world a certain line, its *world line*. If we could describe the world lines of all substantial points of the universe (atoms, light waves), we should have the history of the whole universe, past and future. Every phenomenon comes to be the intersection of the world lines of substantial points and every observation made by any of us, whether a child or a savant, is knowledge of the intersection of world lines. If we have a sound wave produced by the professor's organ of speech which strikes on the hearer's tympanum, or a light wave produced by a flash of lightning which strikes on the retina—each of these events can be regarded as the intersection of two world lines; the world line described by the wave and the world line described by the one or other

organ of our body. This applies both to the observations of the astronomer and to the experiments of physicists. Electrical measurements, determinations of temperature, weight, pressure, and so on, in the end are reducible to finding the coincidence of some pointer with some division of a scale. If many things that we know about the external world are not of this character, if, for example, in weighing a letter on the palm of our hand, we have only a certain muscular sensation, yet in that instance, too, any more accurate determination must be found by a comparison with accurate measurements based on coincidences. Every coincidence which we observe may be regarded as an intersection of world lines. Thus, all natural science is reduced to the study of the intersection of world lines. It is this which gives great importance to Minkowski's idea of the world as a four-dimensional manifold each of whose elements (elementary events or point events) is defined by four numbers. Henceforth, physics must become one of the chapters of the general study of continuous manifolds of four dimensions created by the mathematicians of the nineteenth century.

7. THE MATHEMATICAL EXPRESSION OF THE SPECIAL THEORY OF RELATIVITY

In chapter III we saw that the geometry of surfaces and spaces might equally correctly be considered as a part of the subject of manifolds of two and three dimensions. Different surfaces are characterised by the expression for the line element

$$ds^2 = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2,$$

or by that group of transformations which leaves the magnitude of the line element invariant. Similarly, to the spaces of Euclid, Lobachevski and Riemann, there correspond a characteristic expression for the distance between points (either at a finite distance or infinitely near) and a characteristic group of transformations.

If physics is a new chapter in the theory of manifolds, where the number of dimensions is four, then it is natural to look for similar characteristic expressions and groups of transformations in the manifold which it studies. This manifold is the world. The special theory of Relativity in the generalised form given to it by Minkowski answers this question. Its fundamental principle which Minkowski proposed to call the *postulate of an absolute world or the world postulate*, asserts the invariance of *all* the laws

of nature in relation to linear (affine) transformations for which the function

$$x^2 + y^2 + z^2 - c^2 t^2$$

(which we will call I) is invariant.

Group L of these transformations characterises physics—the study of the laws of nature—just as the group of transformations for which $x^2 + y^2 + z^2$ is invariant characterise the geometry of Euclidean space.

We can formulate the principle of the special theory of Relativity or the world postulate in another way. The world is an affine four-dimensional manifold, whose measure relations (metric) are defined by the function I . All the laws of nature ought to follow from this definition of the world just as the properties of surfaces are deduced from the form of the line element.

$$ds^2 = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2.$$

In the geometry of surfaces, as we pointed out in chapter III, great importance attaches to the so-called geodesic lines which are obtained as solutions of the problem of the shortest distance between two points A and B of the surface. The distance between these two points is expressed by the integral $\int_A^B ds$. The problem of finding the shortest distance is one of the calculus

of variations. The variation of the integral should vanish

$$\delta \int_A^B ds = 0.$$

For flat surfaces ($ds^2 = dx^2 + dy^2$) we obtain lines whose equations are

$$\begin{aligned} x &= ap + b, \\ y &= a_1p + b_1, \end{aligned}$$

where p is a parameter; that is, we obtain straight lines. Straight lines are also obtained as the solution for the analogous problem in the case of Euclidean space for which

$$ds^2 = dx^2 + dy^2 + dz^2.$$

If we solve the same problem of finding the geodesic or extreme lines for the world (*i.e.* a four-dimensional manifold in which we have a corresponding expression for the line element), then we find that a geodesic line of the world is given by

$$\begin{aligned} x &= ap + b, \\ y &= a_1p + b_1, \\ z &= a_2p + b_2, \\ t &= a_3p + b_3, \end{aligned}$$

when p likewise is an auxiliary parameter. By eliminating the parameter, it is possible to express x , y and z as linear functions of the time t . But such a world line corresponds to motion in

a straight line with constant velocity, *i.e.* to the motion of a particle according to Newton's first law, 'unless it is compelled to change its state of motion by impressed forces.'

The first law of classical mechanics can then be stated as follows: The world line of a material particle not acted on by any force is a geodesic line given by the equation $\delta \int ds = 0$.

In the next chapter we shall see the importance of stating the law of inertia in this way.

The special (null) lines of the propagation of light are equations which are invariant for transformations of the group L just as those of a freely moving particle are. Similarly, on the basis of the special theory of Relativity all the laws of nature ought to be invariant in relation to the transformations of group L . Hence, the expression of these laws ought to be found as a particular case of more general equations which are invariant for all linear transformations. Such equations have been found in the mechanics of a rigid body by means of vectors. In the mechanics of elastic bodies and in electrodynamics we meet with more general expressions *tensors* (deformation tensor, stress tensor), and the laws of these parts of mathematical physics have been expressed by means of tensors in a three-dimensional space.

The problem of expressing the laws of mechanics and electrodynamics in tensor form was partly solved by Einstein, but Minkowski obtained a particularly simple solution, by introducing in place of the time t a new variable $\tau = ict$. ($i = \sqrt{-1}$.)

The invariant of group L is then expressed in the form $x^2 + y^2 + z^2 + \tau^2$, i.e. in a form precisely analogous to that of the invariant of Euclidean geometry. Using the language of the theory of manifolds we can state the final result of the special theory of Relativity as follows:

The world is a Euclidean manifold of four dimensions. Its line element is expressed in the quadratic differential form

$$ds^2 = dx^2 + dy^2 + dz^2 + d\tau^2.$$

We call to mind now (see chapter III) that the line element of a plane in a Euclidean manifold of two dimensions must be represented, if we use curvilinear coordinates in the more general form

$$ds^2 = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2,$$

in which the coefficients g (functions of u and v) have to satisfy a certain differential equation expressing the fact that the Gaussian curvature of the manifold is zero.

If the line element of the world of the special

theory of Relativity is transformed to new variables, it similarly will have the form

$$g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 \\ + 2g_{12}dx_1dx_2 + \dots + 2g_{34}dx_3dx_4,$$

which contains ten different coefficients g_{11} , g_{22} , g_{33} , g_{44} , g_{12} , g_{13} , g_{14} , g_{23} , g_{24} , g_{34} .

According to the theory of manifolds of many dimensions, whose foundations were laid by Riemann, the necessary and sufficient condition that a four-dimensional manifold should be Euclidean is that a certain tensor of the fourth degree (usually called 'the Riemann Christoffel tensor') should vanish. This condition can be replaced by twenty independent equations, just as in mechanics a vector equation can be replaced by three equations.

The geometry of the manifold is not altered by an alteration in the form of the line element if these twenty conditions are observed; in particular, the law of inertia is essentially the same as that of classical mechanics. But the special theory of Relativity differs from classical mechanics by introducing a general idea of the world and stating the law of inertia in a new form. The introduction of this general idea, an inevitable consequence of the group of transformations L has made it possible, as we shall see in the next chapter, to generalise the prin-

ciple of Relativity and create a new theory of gravitation. Thus the special theory of Relativity was a necessary step to the general one. On the other hand, it did not lose its importance after the general theory was formulated because its formulae and theorems are first approximations to the formulae and theorems of the general theory of Relativity.

CHAPTER V

THE GENERAL THEORY OF RELATIVITY

This is the most important result in connection with the theory of gravitation since Newton's days. Einstein's reasoning is the result of one of the highest achievements of human mind.

J. J. THOMSON.

I. ABSOLUTE MOTION AND INERTIA

Newton based classical mechanics on the notions of absolute time and absolute space. The special theory of Relativity got rid of the former and at the same time fused time and space into the more general notion of a 'world.' But it did not touch on the question of absolute space and absolute motion and left them in the same position in which they were placed in the "Principia."

In his well-known scholium to Definition 8 Newton says: 'It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of bodies from the apparent; because the parts of that immovable space in which bodies truly are moved do not come under the observation of our senses.'

The principle of Relativity of classical mechanics formulated by Newton asserts in addition that it is quite impossible to determine whether or not the system is moving uniformly in a straight line. It is impossible to distinguish

rest from motion by any experiments made within the system. Let us call to mind that picture of the phenomena which takes place in the cabin of a ship, sketched by Galileo in his "Dialogue." The impossibility of discovering uniform motion in a straight line shews the impossibility of speaking of a definite place in Newton's absolute space.

A body which to one observer inseparably connected with it seems to be at rest appears to another observer to be moving uniformly in a straight line. If for the first observer the body defines a certain place in absolute space, the second has a full right to deny this. Thus in absolute space there is no place which it is possible to mark and define by physical means. But if definite places in absolute space have no physical reality, nevertheless absolute space exists and its reality is seen in rotatory or accelerated motion. To the above quoted words about the difficulty of distinguishing the true motions of bodies Newton adds: 'Yet the matter is not altogether desperate.' The way to ascertain true motions lies in the study of the forces which are at the basis of true motions as their active causes. In particular, they appear in rotation in the form of centrifugal force. 'The effects,' writes Newton, 'which distinguish absolute from relative motion, are the forces of receding from the axis

of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion.'

In this way Newton saw in the phenomenon of the force of inertia a proof of the existence of absolute motion. According to him the physical reality of absolute space was demonstrated by the resistance which bodies shew to a change of velocity by the forces of inertia which then appear and which we experience when the break is put on a tram or a railway carriage goes round a curve. It is seen in the flattening of the earth, in Foucault's experiment of the change of direction of a swinging pendulum, in the diminution of the force of gravity at the equator, etc.

Let us consider the following imaginary case. Suppose that in the universe there are two fluid bodies S_1 and S_2 made of the same material, and of the same size—at such a distance from one another that the action of gravitation of one body on the other is negligible. An observer on S_1 , which he takes to be at rest, discovers that S_2 is revolving; but conversely for an observer on S_2 , which he takes to be at rest, S_1 is revolving. We have an instance of relative motion precisely similar to the case of the uniform motion in a straight line of one system relative to another. But then, as in the last example, no experiments

conducted in one system can decide which system is in motion. In the case we have considered of S_1 and S_2 measurements on these bodies can shew that S_1 is a sphere and S_2 an ellipsoid. What is the explanation of this difference? S_1 cannot be considered to be the cause of the flattening of S_2 because the bodies are in precisely the same relation to each other. Mechanics based on Newton's principles concludes that the change of the shape of S_2 is due to its absolute motion, *i.e.* it finds the cause in absolute space. But this hypothesis of absolute space, which is justified only by the facts which it is introduced to explain, is not satisfying to a critical mind. In chapter II we quoted the passages in Berkeley's works where he controverts Newton's arguments. We mentioned that Kant in the first part of his "Principles of Metaphysics" decidedly insists on the kinematic relativity of all kinds of motion without exception, and only in the second part introduces new arguments to defend absolute space, which, nevertheless, do not appear more convincing than those of Newton. When in the presence of a medium we see strange phenomena, then, before recognising as the cause of them a new 'psychical' force of the medium, it is indispensable to ascertain for certain whether it is impossible to explain these phenomena except by creating new forces *ad hoc*. Starting

from such a theoretically recognised principle Mach, as we shewed in chapter II, put forward another hypothesis in opposition to the hypothesis of absolute space and shewed that it was possible to find a real physical cause. If, besides the bodies S_1 and S_2 , no other bodies existed in the universe then the phenomenon of the flattening of S_2 would actually be completely inexplicable. But in actual fact we can never conduct experiments on two bodies floating in isolation in space. In the actual world, in our cosmos we see a crowd of constellations, an immensely great number of distant masses. Each world body is surrounded by numberless other bodies completely remote from it and moving so slowly that they can be considered as a firm envelope in which the body we are observing is enclosed. According to Mach these distant masses appear to be the cause of the forces of inertia which are developed in rotation and produce those phenomena which Newton's mechanics explained by means of absolute motion. In this way Mach had already connected the problem of absolute motion, or, in other words, of the general principle of Relativity which denies every kind of absolute motion, with the forces of inertia and raised the question whether it is possible to reduce inertia to the mutual action of the body and distant masses. This question, nevertheless,

still appears to be unsolved. 'With Mach,' writes Einstein in February 1921, 'I feel that an affirmative answer is imperative, but for the time being nothing can be proved. Not until a dynamical investigation of the large systems of fixed stars has been performed from the point of view of the limits of validity of the Newtonian law of gravitation for immense regions of space will it perhaps be possible to obtain eventually an exact basis for the solution of this fascinating question.'*

Until this question is settled, so long as there is no possibility of making experiments to shew that in the neighbourhood of very large and swiftly revolving masses centrifugal forces are developed on bodies at rest, which thus would appear to be the result not of absolute space, but of the action of other masses,† the hypothesis of absolute space might be replaced by the other hypothesis.

The hypothesis adopted by Einstein and called by him the General Theory of Relativity both combines the phenomenon of inertia with the 'riddle of gravitation' and at the same time by

* *Nature*, Special number. "Relativity" (1921, February 17th). The question was—Is the spatial extent of the universe infinite?

† B. and T. Friedlander (*Absolute und Relative Bewegung*, 1896) pointed out the possibility of such experiments. [Author's note.]

this means produces a new theory of gravitation.

2. THE PRINCIPLE OF EQUIVALENCE

In his well-known letter to Bentley (1692) Newton says: 'that gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance through a vacuum without the mediation of anything else by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws.'*

Newton leaves it to the reader to decide whether this agent is material or not.

For two centuries since then attempts have continued to be made to explain gravitation, not by means of action at a distance but by pressure or impacts using the facts and results of the theory of elasticity or of hydrodynamics. The development of electro-dynamics naturally led to electro-magnetic theories of gravitation. From Drude's detailed and interesting report in 1897

* The letter, dated Feb. 25, 1692-3, is printed in Nichols' *Illustrations of the Literary History of the eighteenth century* and in the *Correspondence of Richard Bentley, D.D.*, vol. 1. p. 70.

("Annalen der Physik und Chemie") it is possible to form a notion of the two hundred hypotheses of various kinds by which people sought to explain the phenomenon of gravitation. Attempts to adopt the formulae of the special theory of Relativity to gravitation were made by Lorentz, Minkowski and Poincaré. Under the influence of the special theory new theories of gravitation were propounded by Abraham and Nordström. But in all the numerous attempts to understand the nature of gravitation no proper attention was paid to that property of gravitation which sharply distinguishes it, for example, from electro-magnetic force, to a fact already discovered by Benedetti. Bodies moving solely under the action of gravitation fall in an identical manner with an acceleration independent of their substance or their physical condition (chemical composition, temperature, magnetism, etc.).

The ratio or (with a suitable change of the units of measurement) the equality of inertial mass with gravitational mass which flows from this fact, was after that subjected to various experiments (Newton and Bessel with a swinging pendulum, Eötvös (1890) and recently Pekár (1919) by means of a torsion balance).

This numerical equality led Einstein to the hypothesis that the two phenomena—inertia and gravitation—had one common origin. It is im-

possible not to see the analogy of this hypothesis with Newton's hypothesis of the identity of the cause of bodies falling and the motion of the earth round the sun. Einstein supports his hypothesis by a mental experiment shewing that as a consequence of the equality of inertial mass with gravitational mass the acceleration of a system cannot be distinguished by an observer who moves with this system from motion produced by gravitational force (*the principle of equivalence*).

Let us suppose an observer in a closed box who suddenly notices that all bodies begin to fall with one and the same acceleration towards the bottom of the box. He may suppose that his box has come into the sphere of gravitation of some new body directly under the box, and that in consequence of this the box remains where it is, but the bodies in it begin to fall. But he may make a different hypothesis, namely, that someone has begun to drag the box upwards with a constant force producing a uniform acceleration; then the falling of the bodies downwards is explained as being due to their inertia. The possibility of such two equally reasonable hypotheses is a consequence of the law of the equality of gravitational mass with inertial mass. This possibility of two different explanations of the same phenomenon is evidently a case of the

general principle of Relativity. Just as it is not possible to determine by any experiment whether the system is at rest or moving uniformly in a straight line, so in the example under consideration the observer cannot determine by any experiment whether the box is at rest and the bodies in it are falling or whether on the contrary the floor of the box is moving towards the bodies which remain at rest.

Extending the equivalence of uniformly accelerated motion and motion under the influence of a constant gravitational force to all observed cases of curvilinear and accelerated motion we may say that at every point of the universe every kind of accelerated and curvilinear motion observed by us in a free body, *i.e.* a body which is not under the action of an electro-magnetic field nor of material impulses, can be regarded as revealing a certain directing field. At the same time it is a gravitational field. But if this field does not merely produce uniform motion in a straight line then that metrical world in which the world lines of free bodies are straight (which is the basis of the special theory of Relativity according to which the world is a Euclidean manifold of four dimensions) must be changed for a more general one. Such a generalisation must at the same time change the special principle of Relativity into a general principle which

does not, like the special principle, select uniform motion in a straight line from the general aggregate of all possible motions.

Such shortly stated are the considerations which lead to the fundamental hypotheses of the general theory of Relativity.

I. The world is a non-Euclidean manifold of four dimensions whose line element is defined by the formula

$$ds^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 \\ + 2g_{12}dx_1dx_2 + 2g_{13}dx_1dx_3 + \dots$$

in which $g_{11}, g_{12} \dots$ are ten functions of the four coordinates of an element of the world (point events). The metrical properties of the world (the mutual relations of measuring rods and clocks) are different at separate points; the forces of inertia depend on these metrical properties.

II. The ten functions g_{ik} which define the metric of the world also depend on the distribution of matter in space-time and determine the gravitational field. They may be called gravitational potentials. The connection which, through this twofold significance of the functions g_{ik} , is established between the metrical world and gravitation, answers the question, which Riemann propounded, about 'those acting and binding forces' which are the foundation of his definition of measurement.

From this point of view Lobachevski's apparently paradoxical phrase about 'the forces which produce everything—motion, velocity, time, even distance and angles' attains an historical importance.

III. The laws of the world should be expressed in an invariant form in relation to the group of transformations

$$\left. \begin{aligned} x_1' &= \phi_1(x_1, x_2, x_3, x_4) \\ x_2' &= \phi_2(x_1, x_2, x_3, x_4) \\ x_3' &= \phi_3(x_1, x_2, x_3, x_4) \\ x_4' &= \phi_4(x_1, x_2, x_3, x_4) \end{aligned} \right\} (E),$$

where the arbitrary functions ϕ have, nevertheless, to satisfy certain mathematical conditions (continuity, differentiability).

The laws of the world being invariant in relation to the group E will be invariant in relation to the subgroup of transformations which consist of all possible movements. It is in this invariance in relation to all movement that the mathematical foundation of the general principle of the relativity of motion consists. Such are the fundamental hypotheses of the general theory of Relativity which make it possible to construct a mathematical theory of gravitation which differs from that of Newton.

The fundamental assumption of the infinitesimal calculus first of all make it possible to

convert the group of transformation E into linear transformations between the differentials of the new and the old variables. This conversion is of the highest importance; for it makes it possible to apply the theory of tensors and to express the laws of nature in tensor form. The fundamental problem of the theory of gravitation appears then to be to discover the equations of the gravitational field (*i.e.* the equations which the functions g_{ik} satisfy) in such tensor form. In the particular case of the special theory of Relativity, these functions satisfy the twenty equations of which we spoke in chapter IV. In the general case of a 'clear' gravitational field obtained in the absence of matter (by matter Einstein means everything other than gravitational fields, *i.e.* he includes the electro-magnetic field in this concept) Einstein, guided by analogy with Laplace's equation, which in Newton's theory is satisfied by the gravitational potential, and making use of the results of Riemann's theory of manifolds, finds ten equations of the gravitational field. The same method leads him to the equations which connect the gravitational potentials with the constituents of the tensor which characterises matter (the impulse-energy tensor).

These equations consist of a generalisation of Poisson's equation which connects Newton's potential with the density of matter. These and

other equations were discovered by Einstein by analogy and induction. The same equations almost simultaneously were deduced by means of the calculus of variations (Hamiltonian principle) by Hilbert who took as the foundation of his deductions on the one hand the fundamental ideas due to Einstein that the laws of nature were invariant and on the other hand Mie's theory of matter according to which matter is regarded as an electrical phenomenon and is mathematically characterised by the four components of a certain vector in a four-dimensional space-time continuum. Thus Hilbert's result made it possible with the help of a general world function to combine in one formula the theory of gravitation and electro-magnetism and in this way shewed the possibility of that further generalisation of Einstein's which was developed (following Weyl) in his theory of electro-magnetic phenomena.* As we shall see in the next chapter, Weyl's theory also constitutes a new step in the development of scientific thought. It attaches to the principle of the Relativity of motion the new principle of the Relativity of magnitude and alters one of the hypotheses which lie at the base of Riemann's theory of manifolds for another and more general one.

* See Einstein, "Zur allgemeinen Relativitätstheorie." *Sitz. der Preuss. Acad. der Wiss.* (1923), p. 32.

3. THE GENERAL THEORY OF RELATIVITY IN THE LIGHT OF THE LATEST DISCOVERIES IN PHYSICS AND ASTRONOMY

The task of obtaining the functions g_{ik} from the equations of gravitation presents great mathematical difficulties. But in one case, if we make certain simplifying assumptions, the equations can be solved. Such a case is a gravitational field produced by a spherical mass at rest. If we assume that all the g_{ik} are independent of the time (statical field) and that the gravitational field has spherical symmetry, ds^2 takes a moderately simple form and the corresponding equations of a geodesic line makes it possible to determine the motion of a free particle, *i.e.* (according to Einstein's theory) a particle moving only under the influence of the gravitational field produced by the central mass. If we proceed to solve for the g_{ik} 's with our simplifying hypothesis we obtain as a first approximation Kepler's laws of planetary motion, *i.e.* an agreement with Newton's law of attraction. The second approximation gives the result that the major axis of the ellipse ought at each revolution to move in the direction in which the planet moves and also the formula for calculating the angle of revolution of the axis in terms of the major axis and the eccentricity of the ellipse. In chapter II we mentioned the

only phenomenon which could not be explained on the basis of the formula of celestial mechanics—the movement of the perihelion of Mercury. In a century the perihelion of Mercury changes by $547''$. The theory of perturbations explains the part due to the influence of other planets (mainly Venus), but there remained an unexplained residue of $41.24''$ (Leverrier in 1845).

To explain this residuary shift various hypotheses were put forward (a change of the exponent in Newton's law, contraction of the sun, an application of the electro-magnetic law, the zodiacal light), but these hypotheses either did not produce complete agreement with the facts or produced it only on the basis of numerical hypotheses selected *ad hoc*. Einstein's formula gave $42.89''$ for the displacement of Mercury in a century. If, therefore, the deduction of Kepler's laws from a new theory, based on different foundations from those of Newton cannot fail to call attention to Einstein's theory, the coincidence of Einstein's figure with that derived from observation has all the greater significance.

Einstein deduced another consequence from his theory. From the principle of the equivalence of a gravitational field with the field of inertia obtained by the accelerated motion of a system

it follows that as in the second case a ray of light cannot pass through the accelerated system without being bent from a straight line, so a gravitational field must similarly bend a ray of light. Calculation shews that the magnitude of the bending of the ray of light from a star which passes near the sun is within the limits of astronomical observation for stars whose ray passes near the disc of the sun. But such a bending can only be observed during an eclipse. Two English expeditions, under Eddington and Crommelin, to observe the solar eclipse of 29th May 1919 on the west of Africa (the island of Principe) and Brazil (the village of Sobral) obtained results very near to the bending deduced from Einstein's theory.*

Finally, a third possibility of experimental proof arises from the influence of a gravitational field on the change of time. Clocks should go at different rates depending on gravitation. An atom which produces periodic movements is such a clock.

The spectral lines of any object on the sun or other equally great heavenly body ought to correspond with a smaller number of oscillations of the same object on the earth. The spectral lines, which come to us from the sun or stars

* These results were confirmed by observations made during the solar eclipse of September, 1922.

ought therefore to be shifted towards the red end of the spectrum.

According to Einstein's calculations, the proportionate shift is expressed by the small fraction

$\frac{2}{10^6}$ and for lines at the violet end of the spectrum

would be 0.008 Angström (an Angström is 10^{-8} cm.). Although these shifts are very small, nevertheless they are within the limits of measurement of modern spectroscopy. But the results of scientists in America (the Mount Wilson observatory), Germany (Grebe and Bachem), and France (Pérot, Fabry and Buisson) cannot be said to be a conclusive confirmation of Einstein's hypothesis. The confirmation of this hypothesis would have great cosmological significance, as it would make it possible from the amount of the shift of a line of the same element in the spectra of different stars to deduce conclusions as to the relative magnitude of these stars.*

Such are the deductions from Einstein's theory which already in the present conditions of physical and astronomical technique may serve to verify it experimentally. But it makes such a wide and profound alteration in our outlook on the

* The Einstein shift has been confirmed by Evershed (*The Observatory* (1923), pp. 299, 343) and St John (*Monthly Notices Royal Astronomical Society*, December, 1923).

fundamental questions of physics that its further development must inevitably lead to the putting and the solution of new physical problems which till recently had seemed to us impossible. Thanks to it a way may be found to solve the problem of the forces which act within the atom, of the immense store of internal atomic energy. On the other hand, it has already raised the question of the structure of the universe as a whole.

Astronomers (in particular H. von Seeliger) had already, before Einstein's theory, pointed out a contradiction between Newton's laws and the hypothesis of an unlimited space, and they sought a way out of this contradiction by supposing that the universe is not Euclidean space, but that it, like the surface of a sphere, is unbounded but finite.

In chapter III we mentioned that this question already occupied Riemann who was the first to point out that from the boundlessness of space (which in itself has greater empirical credibility than any other external experience) it does not follow that space is infinite. He was the first to point out the possibility of creating the geometry of a space which was unbounded but not infinite, the space which is named after him. But he added 'that the questions of the infinitely great are idle questions.' But to-day both astronomical

considerations* and Einstein's theory lead us to these idle questions.

Basing himself on the extremely important cosmological fact that the relative velocities of the stars are excessively small, Einstein finds it possible to separate out a world of four dimensions in space and time.† According to Einstein's formula this space is spherical. Its geodesic lines are circles and their radii can be calculated from the formula of the theory of Relativity. De Sitter, on the basis of astronomical investigations as to the mean density of the universe, calculates it to be approximately 20 million light years and, consequently, the length of a geodesic line to be 100 million light years. A ray of light moving along this geodesic line returns again to the source of light after 100 million years. Just as great circles on a sphere which come from the north pole intersect again at the south pole, so in spherical space each star may have its 'anti sun'; and possibly a part of the stars which we see in the sky may merely be such 'anti suns.'

Such are the wide cosmological horizons which the general theory of Relativity opens before us.

* These are set out in detail in the popular articles of Harzer and Swarzschild. [Author's note.]

† See *The Principle of Relativity*, by A. Einstein and others, p. 177 *et seq.*

But independently of these problems which it suggests and of those phenomena which it explains, the immense importance of the theory of Relativity created by Einstein consists primarily in the grandeur of aim he has set before him, namely, to construct physics as a mathematical theory of a four-dimensional manifold. But in stating this problem of the creation of the geometry of the world and in extending to all physics the fundamental idea of the geometry (the idea of invariance in relation to group transformations) the theory of Relativity offers new points of view also for the solution of the problem of space. It finally gets rid of the idea of a geometry independent of physics and establishes that connection between the two sciences to which, as we said at the end of chapter II, both the investigations of mathematicians and the work of the psychophysicologists nevertheless led us.

It appears undoubtedly as the starting point of a new philosophical view of the world. Its conceptions to-day touch on those questions of space, time and causation which have always appeared to be the most important questions of philosophy, and also the question of the distinction between the real and the accidental, of finding an unchanging reality beyond the world of our changing sensations, which appears to be the fundamental question of philosophy.

Henceforth no future philosophical point of view will be able to avoid the theory of Relativity. In the next chapter we will speak of the relation of some philosophical schools to it at the present time.

CHAPTER VI

THE PHILOSOPHIC MEANING OF THE THEORY OF RELATIVITY

“Reality, how shall I seek or know?
What shall I do and whither shall I go
Fleeing from fear and sorrow without sleep?
Thought only hath my safety in her keep,
Thought only gives me calm, and rest most deep.”

From the Russian of N. A. VASILIEV, *Longing for Eternity*.

In the history of human thought, we meet again and again the famous struggle of two philosophic points of view; the one is contented with the study of phenomena, recognising that our sensations are the only reality that we can reach, the other strives in the words of the poet we have quoted ‘to tear the veil from the false pale ghost of seeming’ and find the hidden reality of which phenomena seem to be the reflection. This strife has taken different forms throughout the ages; and in the opening chapters of this book we dwelt in detail on certain moments of this struggle which were closely connected with questions of space, time and motion. This may be briefly summed up as follows.

The naïve realism of the Ionian school recognised water, air, fire as reality. The phenomenalism of Protagoras declared that human thought was the only measure of all things. To both these the Pythagorean school opposed its doctrine that all knowledge was connected with

number; nothing can be understood without number. This first mathematical philosophy was developed into the atomistic theories of Democritus and Plato's doctrine of ideas. Both these see the truth, not in sensations which are always changeable, but behind these sensations they seek the hidden true reality. Atoms moving in empty space are for the first the explanation of all the phenomena of the universe, while for Plato all phenomena are only shadows on the wall of the cave thrown off by bodies which are unseen by the prisoners in the cave. Both agree as to the necessity of finding a reasonable basis of all being, a 'logos,' and both see this 'logos' in mathematical truths, in arithmetic and geometry. There is no doubt about this in the case of Plato; it is proved by the deep reverence for mathematics with which he inspired his school. For Democritus no less, who was a distinguished mathematician, atoms were undoubtedly the answer to the problems of divisibility and indivisibility, continuity and discontinuity which were raised in the Pythagorean school by the discovery of an irrational number.

But the reality which was discovered by the Pythagorean philosophy of Plato and Democritus cannot satisfy sceptical minds. We know from Zeno's 'aporiai' what depths the dialectics of Greek thinkers can reach in problems of space,

time, and motion. Zeno appeared as the representative of the metaphysic directly opposed to that of the Pythagoreans. Zeno's metaphysic demonstrated his reality of the 'One.' It seems however that the phenomenological point of view which Protagoras had developed, also found supporters in that period at the end of the fifth and beginning of the fourth century B.C., which was so important in the history of thought. It was this time, as we know from the life of Democritus, that saw the beginning of experimental science, the study of phenomena. Aristotle was a disciple both of Plato and of doctors representing the school of experimental science; and from this point of view Aristotle's philosophy may be regarded as the synthesis which united experimental science and observation with the idealistic philosophy of his predecessors. This synthesis satisfied human thought for many centuries, but the two opposing tendencies which it appeared to reconcile, again made their appearance in the dispute among Mediaeval Schoolmen between the nominalists and the realists. On the problem of space and time Aristotle already adopts the experimental standpoint as far as was then possible. At a time when for Democritus, space was empty not-being, and for Plato it was something intermediate between ideas and matter, Aristotle replaces the problem

of space by that of place and his statement of this problem, as we have seen in chapter I, leads to that of absolute and relative motion. The obscurity of the passages dealing with this subject in the "Physics" and Aristotle's other works was the subject of the animated disputes between his commentators, of which we have given some details in chapter I. The interpretations upheld by Averroes and his school were the chief support of the opponents of Copernicus and Galileo.

The Renaissance brought about the re-birth not only of Platonism but of the atomistic philosophy of Democritus. Under the influence of Platonism the doctrine of space took a mystical form both in the Italian (Patricio, Campanella) and the English schools of natural philosophy. This mystical doctrine of space as spiritualised matter is given clearer expression than anywhere else in Henry More's "Enchyridion" which appeared only thirteen years before Newton's immortal book. Unquestionably the atmosphere of English Platonism influenced Newton's metaphysical views, his formulation of the concepts of absolute space and time, his doctrine of the immaterial nature of force. At the same time, however, Newton was one of the greatest founders of mathematical physics and in particular of the mechanical view of the universe. It is this that constitutes his immortal service. The great im-

portance of the classical mechanics in its application to the macrocosm and the microcosm naturally makes people wish to construct a basis for the metaphysical postulates of the "Principia." The most important of these bases are those of Euler—one of the greatest mathematicians of the eighteenth century—and of Kant whose philosophy had an immense influence on the whole of the subsequent history of philosophy. Euler and Kant could not fail to take into account in this connection, the objections to the metaphysical views of space which were advanced by the Irish philosopher Berkeley. Berkeley, who was the first in the new philosophy to renew the tradition of Protagoras, was the representative of the phenomenalism which followed from it. This denied the very existence of matter. Berkeley's arguments against absolute space and absolute motion explained in his chief work, "A Treatise concerning the Principles of Human Knowledge," and also in a work, "De Motu," specially devoted to problems of motion, have not even yet been given their right place. Berkeley's arguments had practically no influence on mathematical physics or even on philosophy. The former was entirely under the influence of the brilliant success of the new mechanical view of the universe. In philosophy the struggle between the phenomenological views of Berkeley and

Hume on the one hand and the metaphysical systems of Berkeley himself,* Descartes, Spinoza and Leibniz, was settled for a time by the new synthetic philosophy of Kant. The recognition of the complete determinism of events is reconciled in this system with the recognition of the 'thing in itself.'

Kant's doctrine of space and time influences his whole philosophy. His general philosophical views are always dependent on his view of these problems. So his relation to the problem of absolute space is closely connected with his doctrine of space. Kant's doctrine in the "Critique of Pure Reason" had immense influence both on the philosophy and the science of the nineteenth century. Here space was recognised as a subjective form of our perception with which all our sensations must necessarily be invested; geometrical truths are not truths of experience, but they too, like space, are given *a priori*, as necessary conditions of any experience. Geometry, the science of pure space, and algebra, the science of pure time, are purely formal sciences, entirely cut off from the basis of experience and having no connection with the material provided by our sensations.

* Berkeley developed an idealistic system closely akin to Platonism, more especially in his later works, *e.g.* *Siris*. [Author's note.]

Kant's idealistic doctrine was further developed by Fichte and Schelling. Schelling, for example, thought it possible to deduce the three dimensions of our space from the nature of our soul.

In the view that space and time were independent of phenomena taking place in the world, the doctrines of the idealistic philosophy of the nineteenth century were at one with those of the materialistic philosophy. 'In Euclidean space and uniformly flowing time the motion of atoms proceeds, and these motions explain all phenomena'—'except consciousness,' the idealists added. 'Consciousness included,' retorted the materialists. The first breach in this construction, which seemed almost finished, was made by the immortal creators of the non-Euclidean geometry—Gauss, Lobachevski and Riemann first threw doubts on the sharp distinction between our presentations of space and our other sensations and first considered space with the eyes of a physicist. In chapter III we quoted those passages from the works of Lobachevski and Riemann in which they expressed their views on the connection between forces and the metrical properties of space. As if to develop these ideas, which were merely brilliant suggestions, Clifford constructed a theory by which motion and matter are mere manifestations of the curvature of space and nothing further. The

philosophers and mathematicians who studied the problem of the origin of geometrical axioms, arrived at the same conclusion as to the connection between geometry and physics. Berkeley and Ueberweg had demonstrated the connection between the axioms of geometry and the properties of the motion of a solid body. The mathematical formulation of this connection was given by Helmholtz and Sophus Lie in the theory of the continuous groups of the transformations of space considered as a three-dimensional manifold. These works, as well as the researches of Gauss, Grassmann and Riemann, made mathematicians increasingly familiar with the idea of space as a particular case of a many-dimensional manifold, and with the necessity of constructing a theory of such manifolds. Such a theory was in fact constructed on the foundations given by Riemann, and the general theory of relativity could use it as a necessary instrument.

On the other hand, the works of the psychophysicists and the English empirical philosophy, developing the ideas and investigations first initiated by Berkeley, led to the conviction that our space presentations are achieved gradually both in the individual and the race, as the result of interaction between our psychophysiological organisation and the environment, *i.e.* physical phenomena.

In this way both non-Euclidean geometry and psychophysiology shattered the belief in Kant's doctrine of the transcendental nature of the axioms of geometry and of the independence of our presentations of space and time from experiment and observation.

Among the important representatives of mathematical physics in the second half of the nineteenth century both Helmholtz and Poincaré gave utterance to views tending in this direction. No one however in the nineteenth century expressed his phenomenological views so clearly as Mach, who worked out these ideas in detail both as to the axioms of geometry and the foundations of mechanics. His short paper in 1872, "The Principle of the Conservation of Work," is certainly one of the most remarkable of the works of scientific philosophy. Here Mach, first after Berkeley, expressed doubt of Newton's postulates and proposed his own hypothesis to explain the phenomena observed in rotation which Newton, Euler and Kant had used as a proof of absolute space. No one more than Mach helped to create that mental atmosphere thanks to which views on the theory of relativity, however paradoxical they might appear, aroused all the more attention. In his obituary notice of Mach, Einstein writes:

'Apparently Mach would have arrived at the

theory of relativity, if at the time when his mind still had the freshness of youth, the question of the constancy of the velocity of light had already engaged the attention of the physicists.*

In actual fact, the anticipations of Lobachevski, Riemann and Clifford, and the ideas persistently expressed throughout his life by Mach, only became a developed doctrine and took the form of a mathematical theory after the scope of physical experiments was enlarged and investigations of mechanical masses (which deal with comparatively low velocities) had united with investigations of electro-magnetic and optical phenomena.

All attempts to construct a theory of the latter phenomena on the bases of the classical mechanics and action at a distance, met with no success. In their place the new concept of the 'field' introduced by Faraday became increasingly important; and on the basis of Faraday's views Maxwell constructed his mathematical theory of electro-magnetic phenomena. The law of action at a distance is replaced in Maxwell's theory by a system of differential equations, in which there is only one physical constant. This constant is the velocity of light, since the whole theory of light is only one of the chapters of the general theory of electro-magnetic phenomena.

* *Physikalische Zeitschrift*, xvii. (1916), p. 103

The new theory, however, is apparently in contradiction with facts. No mechanical experiments carried out within a system are able to establish whether the system is at rest or in a state of uniform motion in a straight line; similarly none of the different electro-magnetic and optical experiments gave the possibility of solving this problem. At the same time the equations of the electro-magnetic field shewed that the analytic formula which had been given to the principle of relativity in the classical mechanics was not applicable to electrical phenomena. The way out of these contradictions was found by Einstein. He demonstrated that the invariance of the equations—a mathematical fact discovered by Voight and Lorentz—admits of a different interpretation from the one given to it by Lorentz. The contraction which has to be admitted in order to explain Michelson's experiment, is not a contraction taking place in moving bodies as a result of the change of the electro-magnetic forces under the influence of this motion; it results from the fact that the moving observer is bound to apply a different system of measurement of time and length. But in all these transformations the expression

$$s^2 = x^2 + y^2 + z^2 - c^2 t^2$$

remains constant, just as in all transformations of

rectangular coordinate axes, the distance between two points of space and consequently the expression $x^2 + y^2 + z^2$ is constant.

The interdependence of the space coordinates and the time coordinate which is established by the fact that the expression into which all four coordinates enter is a constant, leads to paradoxical consequences. Just as it follows from the constancy of $x^2 + y^2 + z^2$ that the increase of one coordinate leads to the diminution of the others, so from the fact that s^2 is a constant and that the expression $c^2 t^2$ has a minus sign before it, it follows that the increase of the distance between two events in space corresponds to the increase of the time interval between them. Two events happening almost at the same time in almost the same place are in the same mutual relation as two events separated from one another by an immense space and a long interval of time.

The concept of the independence of space and time which is the basis of the concept of absolute time passing without relation to anything external, must be replaced by a new general concept uniting both space and time. Such a general concept—a continuous four-dimensional manifold—was introduced by Minkowski. The universe is a whole made up of elements which are *point events*, each defined by four numbers; each moving observer however divides this whole in

his own way, just as in a paper factory it is possible to stratify the mass of paper in different directions and to make different heaps of sheets of paper from this connected mass. It is not however metaphysical concepts (which even before this had brought time and space together), but the mathematical fact of the invariance of Maxwell's equations and the demonstration of the consequences of this fact by experiments in physics, which leads us to the new conception with its paradoxical consequences. Cannot we find an analogy to this new concept, when we study either the formation of our space presentations or the history of human thought? Cannot these analogies help us to reconcile ourselves to the paradoxical nature of the new conception which enables us to apply the mathematician's theory of manifolds to the study of events happening in space and time?

We spoke in chapter III of the psychophysiological investigations which were initiated by Berkeley. These established the distinction between Euclid's geometrical space to which we have hitherto related all phenomena, and physiological space from which it arose and from which it is sharply distinguished. Even at its nearest approximation physiological space, the space of our immediate experience, is distinguished from the space of science, given us by the increased

scope of our geographical and astronomical knowledge. It was easy for human thought to get over the distinction between left and right, backwards and forwards, not so easy in the case of upwards and downwards, as expressed in Aristotle's doctrine of the natural places of bodies. In order to express the impossibility of something, Sosicles of Corinth says in Herodotus: 'The heaven shall be beneath the earth and the earth float in the air above the heaven before. . . .'* Let us call to mind all that was said against the Antipodes, against people hanging head downwards and the tree-tops turned the wrong way. Even now after many centuries the 'commonsense' of the uneducated man cannot reconcile itself to these facts we take as quite ordinary. Now the recent extension of our knowledge, the introduction into the range of experiment of velocities approaching that of light makes us explain phenomena (Michelson's experiment, the fine structure of the lines of the spectrum) by the fact that the distinction between space and time which seems to us so obvious is only the result of our organisation, and that true reality is a world, an aggregate of *point events* defined by four numbers. What is called the special theory of relativity is the assertion that the laws of the universe are invariants for the group of

* Book v. 92.

transformations which leave invariant the expression s^2 or the differential quadratic form

$$dx^2 + dy^2 + dz^2 - c^2 t^2.$$

The special theory of relativity, however, like the principle of relativity in classical mechanics, refers only to uniform motion in a straight line, and the group L is a group of linear transformations which in its form is very near to the group which characterises Euclidean space and it coincides entirely with it (in the form though not the number of the variables) if we introduce imaginary time as Minkowski did. *The world is then a Euclidean four-dimensional manifold.*

Is it however possible to accept the exclusive position which both the special theory of relativity and the classical mechanics ascribed to uniform motion in a straight line?

Even in remote antiquity thinkers (Zeno, perhaps, and certainly Aristotle) sought for the body which was absolutely at rest amid the chaos of different relative motions. When Protagoras laid down his general principle of man as the measure of all things, of the relativity of all our knowledge, he could not fail to see a still clearer argument in favour of his assertions in the illusions connected with motion. Later, when the question arose as to who was right in the question of the immobility of the earth—Aristarchus of Samos or

Ptolemy—this was in reality the problem of the distinction between absolute and relative motion. Newton pointed out the possibility of distinguishing the one from the other. Neither his physical proof, however, nor Kant's metaphysical arguments could give a final answer, and we know that Berkeley in 1721 and Mach in 1872 expressed scepticism as to Newton's idea of absolute space.

Now not only is the question of the relativity of uniform motion in a straight line a particular case of the general problem of the relativity of curved and accelerated motion, but this general problem of the relativity of motion is itself in its turn a particular case of the more general problem of the relativity of changes. The impossibility of distinguishing rest from motion corresponds to the more general problem of what changes might happen without attracting the notice of an observer. The Spanish scholar Balmes (1810–1848) put the question whether humanity would notice if the sun doubled its velocity, provided the same change affected the velocity of the phenomena of the earth and the heavens and this acceleration spread also to us and our ideas. Later Laplace put a similar question in his famous “*Exposition du Système du Monde*.” Laplace saw the special importance of Newton's law of universal attraction in the fact that on the

basis of this law the heavenly bodies would continue to describe exactly similar curves if the dimensions of all the bodies in the universe, their relative distances and their velocities were reduced or increased in exactly the same proportion. 'The universe reduced to the dimensions of atoms would present exactly the same picture to the observer..... The simplicity of the laws of nature allows us to observe and recognise only relations.'

This problem of similar worlds* is only a particular instance of the general problem of the deformation of the universe. We can suppose for example that, as in the hypotheses of Lorentz and FitzGerald, bodies are contracted only in the direction of their motion or are subjected to deformations like those we see in concave mirrors. Whether this deformation would be noticed by an observer (supposing of course that his body and instruments of measurement were subject to deformation under the same law as all the bodies in the universe) is a particularly complicated question, while we are dealing separately with space and time and with physical quantities dependent on space and time.

* This was the subject of an interesting discussion in which Renouvier, Delboeuf, Bouasse and Lechalas took part. The reader can acquaint himself with it in Lechalas' *Étude sur l'Espace et le Temps*, 1896. [Author's note.]

A new light is thrown on this question if we apply to it the new idea of the space-time continuum—Minkowski's world. We said at the end of chapter IV that the introduction of this idea makes it possible for us to assert that all our observations and experiments and all knowledge based on them, in the last resort, resolves itself into the study of the intersection of world lines. Just as in the case of deformations of a surface, the intersection of the lines on the surface does not change—and in deformations of a jelly, the intersection of the lines within the jelly does not change; so in the same way in the case of deformations of a four-dimensional manifold in which the point event defined by the coordinates x_1, x_2, x_3, x_4 changes into the point event x_1', x_2', x_3', x_4' the intersection of the world lines does not change, if the law of the transformations is one and the same for all the point events. These transformations are expressed by the equations

$$x_1' = \phi_1(x_1, x_2, x_3, x_4)$$

$$x_2' = \phi_2(x_1, x_2, x_3, x_4)$$

$$x_3' = \phi_3(x_1, x_2, x_3, x_4)$$

$$x_4' = \phi_4(x_1, x_2, x_3, x_4)$$

in which ϕ_1, ϕ_2, ϕ_3 and ϕ_4 must satisfy certain mathematical conditions (they must be continuous and single valued).

The aggregate of all such transformations clearly forms the group called E in the previous chapter; and if the laws of nature can be expressed in the form of an invariant in relation to the transformations of group E , then the picture of the universe in its changed form will be identical with the picture of the universe before deformation. In this consists that *general principle of relativity* of which the relativity of all movements is a particular instance, since the group of transformations corresponding to motions is only a particular instance of the general group of transformations E .

But the generality of the form of the transformations even in the particular instance of motions of any kind, leads also to the necessity of generalising the form of that invariant differential expression which defines the metric of the world. These are the general considerations which may serve to explain the fundamental proposition of the general theory of relativity.

The world is a four-dimensional manifold, the metric of which is given by the formula

$$ds^2 = \sum_{ik} g_{ik} dx_i dx_k \quad (i, k = 1, 2, 3, 4).$$

The laws of nature must be expressed in the form of an invariant in relation to the group of transformations E which do not change the expression ds^2 .

In chapter v we shewed why the function g_{ik}

may be considered at the same time as quantities characterising the gravitational field, *i.e.* gravitational potentials. Such then are the most important fundamental propositions of the general theory of relativity, which made it possible to consider the theory of gravitation as a chapter in the geometry of four-dimensional manifolds.

We shewed in chapter v the conclusions resulting from the new theory which gave it special importance. It met with great sympathy from mathematicians who saw in it a new proof of the power of mathematical analysis. Eminent contemporary mathematicians, Felix Klein, David Hilbert, Weyl, Levi-Civita, applied their talents to clearing up its foundations and the obscurities of the theory. On the other hand, the novelty of the conceptions and the paradoxical nature of many of the conclusions repel those scientists who are accustomed to deal with concrete fact, with phenomena happening in space and time. This is evidently the cause of the reserved attitude of Lorentz and Lodge and the attacks of Lenard on the general theory of relativity. In answer to these attacks and appeals to commonsense we can point out how often commonsense has deceived mankind. Finally we may quote the remarkable words which Osiander used for an Epigraph to the works of Copernicus. 'There is no necessity that the hypotheses of physics

should be true or even probable. It is sufficient that the results deduced from them by calculation should agree with the observed facts.'

Heated disputes as to the importance of the general theory of relativity will probably interest all thoughtful people for a long time to come. It is interesting to estimate it from the standpoint of the two opposing views of time and space whose struggle we have followed in the beginning of this chapter—the idealistic view originated by the mathematical philosophy of the Pythagoreans and the phenomenalist view whose most notable representative in Greek philosophy was Protagoras.

It would be difficult not to expect a sympathetic attitude towards the general theory of relativity from those followers of the critical idealism, who give special importance to mathematical philosophy. Is not Einstein's theory a striking confirmation of the fundamental idea of Pythagoreanism? And in actual fact, Cassirer, one of the best known representatives of the Marburg school in his "Zur Einsteinschen Relativitätstheorie" (Berlin, 1921), takes the statements of the new theory as completely coinciding with the philosophy of his school.

'The theory of relativity,' he writes, 'teaches us not to take phenomena as they appear to us for truth, *i.e.* for an expression of the final laws

of experience. The subject of physics is not separate properties as they appear to us but only the unity of the laws of nature.* The theory of relativity which destroys the last remnant of the physical existence of space and time and substitutes for them relations between numbers thus, in Cassirer's view gives an important confirmation to the Kantian doctrine of space and time and to the idea specially dear to the Marburg school that invariants are not matter of any kind but only relations or functional interdependencies.

Eddington in his fine work, "Space, Time and Gravitation," develops these ideas in greater detail. The fiction by which he pointedly illustrates his view of the 'nature of things' in the last chapter of the book is of great interest. 'The best comparison I can offer is with a future antiquarian investigation which may be dated about the year 5000 A.D. An interesting find has been made relating to a vanished civilization which flourished about the twentieth century, namely a volume containing a large number of games of chess written out in the obscure symbolism usually adopted for that purpose. The antiquarians, to whom the game was hitherto unknown, manage to discover certain uniformities; and by long research they at last

* P. 56, condensed. Cassirer's book has now been translated into English together with his *Substance and Function*.

succeed in establishing beyond doubt the nature of the moves and rules of the game. But it is obvious that no amount of study of the volume will reveal the time nature either of the participants in the game—the chessmen—or the field of the game—the chessboard. With regard to the former all that is possible is to give arbitrary names distinguishing the chessmen according to their properties; but with regard to the chessboard something more can be stated. The material of the board is unknown, so too are all the shapes of the meshes—whether squares or diamonds; but it is ascertained that the different points of the field are connected with one another by relations of two-dimensional order, and a large number of hypothetical types of chessboard satisfying the relation of order can be constructed. In spite of these gaps in their knowledge, our antiquarians may fairly claim that they thoroughly understand the game of chess. . . . Our knowledge of the nature of things must be like the antiquarians' knowledge of the nature of chessmen, viz. their nature as pawns and pieces in the game, not as carved shapes of wood. . . . The ultimate elements in a theory of the world must be of a nature impossible to define in terms recognizable to the mind.*

The views of Eddington are best summed up

* p. 184.

in his own phrase: 'Give me relations and I will construct matter and motion.'

From this standpoint Eddington defended the new theory of electro-magnetic phenomena which Weyl published in 1918 and dealt with in detail in "Raum-Zeit-Materie."* Just as Einstein's theory defines the metric of the universe by the help of a quadratic form or, to use another expression, by the help of a fundamental tensor whose component parts are the potentials of gravitation; so Weyl connects electro-magnetic phenomena with some linear form or vector.

Just as Einstein's theory is founded on the denial of certain postulates of classical mechanics, so Weyl's theory is based on the removal of an assumption which serves as the base of Riemann's geometry. We saw in chapter III how in his researches in manifolds Riemann introduced a proposition that the unit of length used for the measurement of the manifold was independent of its position in it. In the Einstein-Minkowski theory, it is also assumed that the interval ds^2 and the quantities forming this expression are also measured by one unit, whatever point event we are considering, *i.e.* in whatever point of space or moment of time we undertook the measurement. But is not this premiss an arbitrary supposition? Have we a right to assume

* This is now translated into English.

that supposing we are in Petrograd on the first of March and we wish to carry out a measurement of a certain distance in Kazan on the first of May, the result of the measurement will be identical whether the measuring rod remains in Petrograd till May 1st, and is then transferred to Kazan or whether it is taken to Kazan on March 1st and remains there unused? In other words, does the result of the transfer of the segment depend on the world line on which it takes place? Weyl took as the basis of his theory the premiss that the infinitely small change of the number l which measures the segment in changing from the world point

$$(x_1, x_2, x_3, x_4)$$

to the infinitely near point

$$(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, x_4 + dx_4)$$

is proportional to the number l and is expressed in the linear form

$$\phi_1 dx_1 + \phi_2 dx_2 + \phi_3 dx_3 + \phi_4 dx_4.$$

Just as in Einstein's theory the gravitational field is connected with the fundamental metrical tensor g_{ik} so in Weyl's theory the vector composed of $\phi_1, \phi_2, \phi_3, \phi_4$ defines the electromagnetic field. Thus the world metric is defined by fourteen numbers (ten g_{ik} 's, four ϕ 's) if it is admitted that in nature all manifestations of force can be referred to gravitation and the forces

of electro-magnetism. Weyl proves that the simplest supposition we can make about the function ϕ which is equivalent to the admission that the transfer of the segment is independent of the route taken, leads us to the fundamental equations of electro-dynamics laid down by Maxwell.*

Weyl's theory has not so far been confirmed by any concrete facts and therefore has not as yet found acceptance from physicists. Undoubtedly, however, it represents a step forward on the journey towards the principle of relativity, which human thought has consistently followed.† It introduces a new principle, the principle of the relativity of magnitude in the same way as the general theory of relativity is founded on the principle of the relativity of motion. In its own fundamental character too, it is analogous to Einstein's theory. Both give as their only answer to problems of the nature of gravitation or elec-

* Taking as his basis Mie's theory that matter is an electrical phenomenon and starting out from the principles of the calculus of variations, Hilbert also constructed a mathematical theory of physical phenomena, in which the same 14 numbers that define the physical constitution of the universe appear as the fundamental parameters. [Author's note.]

† Weyl's theory has been developed in a more general way by Eddington (*Mathematical Theory of Relativity*) and Einstein has obtained the field equations which combine gravitation and electro-magnetism ("Zur Allgemeiner Relativitätstheorie," *Sitz. der Preuss. Akad. der Wiss.* (1923), p. 32).

tricity the mathematical relations between world parameters. Descartes' prophecy is fulfilled. 'Sciences in their present condition are hidden by a mask and would only appear in their full beauty if we could tear off this mask. To the mind's eye of any one contemplating the chain of sciences it easily shews itself as a series of numbers.'*

Idealism, revising the fundamental idea of Pythagoras, can see in the mathematical relations between numbers which express the laws of combination of point events—the ultimate elements of the universe—that true reality which is hidden behind our sensations. It can find in the general theory of relativity a new argument against materialism which found reality in atoms and their movements. 'A brain constructed out of differential coefficients of the g 's can scarcely be said to be less adapted to the purposes of thought than one made, say, out of tiny billiard balls.'†

Such are the idealistic deductions from the general theory of relativity. It is however possible to make other deductions, and this is also done. One of the aims of this book has been to shew the atmosphere of scientific thought in which the general theory of relativity arose. Just as

* Descartes, *Œuvres inédits*, by Foucher de Careil. Paris (1859), p. 4.

† Eddington, *Space, Time and Gravitation*, p. 192.

Newton's theory of absolute space grew up in the atmosphere of English Platonism, so the basic ideas of Einstein's philosophy coincide with the ideas of the school of scientific philosophy represented by Mach and Avenarius.

It is not only their relation to problems of inertia and absolute motion that brings Mach and Einstein together. In the preceding chapter we quoted a passage where the creator of the theory of relativity recognises the services of his predecessor. But there are other points where the likeness between the views of Mach and the deductions from the theory of relativity is even more profound.

One of the most important aspects of this theory is undoubtedly the fusion of geometry and physics; in other words, the abolition of the fundamental principle which held the field from the time of Plato and Democritus onwards, the abolition of the distinction between primary and secondary qualities. Mach wrote as early as 1866: 'Physical space which I have in view (it also includes time) is nothing but the interdependence of phenomena. Modern physics which would recognise this interdependence would have no further need of special views on time and space since they would have been already exhausted.'

The persistency with which Mach reiterated

his views on the aim and character of scientific investigations on the necessity of avoiding unreal problems (*Scheinprobleme*) is strikingly expressed in Einstein's physics. 'A conception exists for physics only so far as it is possible to determine whether it is true or not.' Mach seems to have foreseen the importance of the destruction of the dogma of the independence of time and space from one another when he wrote: 'Science gains more from what it ignores than from what it considers.'

We must therefore agree with Petzoldt—one of the best known representatives of Mach's school—when he says in an article on the relation between Mach's ideas and the Theory of Relativity (Preface to Mach's "Mechanics," 7th edition): 'The theory of relativity does not conflict with Mach's views in any of its essential statements. It is the fruit of this thought, which has thrown out deep roots and grown to a mighty tree.'

Mach's views, however, on physics and geometry and the aims of scientific investigation are intimately connected with his philosophy. This consisted in the denial of any other reality apart from our sensations—a train of thought leading us back through Berkeley to Protagoras. Goethe expressed its fundamental idea in the words: 'All natural philosophy is in the last

resort only anthropomorphism. We can at will observe nature, measure, calculate and ponder it, but it always remains *our* impression, our world. Man always remains the measure of things.'

If the general theory of relativity is closely connected with the views of Mach, Cassirer cannot be right in saying that: 'The theory of relativity has only the name in common with the theory of relativistic positivism.'* On the contrary, it is easy to understand the sympathy which the representatives of this school of positivists have extended to the general theory of relativity.

The general theory of relativity, has, as we have seen, points of contact with both the opposing tendencies of philosophy. The representatives of both schools greet it with ardent sympathy. May not this point to the possibility, even in the near future, of a synthesis uniting these views? At the end of the eighteenth century Kant gave a synthesis which brought the determinism of Newton's mathematical philosophy into relation with the idealism of Descartes and Leibniz. This synthesis was for a long time the final stage in the philosophic thought of mankind. Kant's synthesis however does not fit in with the results of non-Euclidean geometry and it is hardly possible to combine it—as many people continue to do—with the results of the

* *Op. cit.* p. 56.

theory of relativity. The new synthesis of the future will not satisfy human thought for ever. The increased scope of physical experiment together with the power of mathematical analysis will open new horizons for the human mind just as Einstein's theory has done. The royal road of human thought will in the future as well as in the past remain equally far from 'the anti-scientific constructions and irresponsible mystical searchings' of to-day's neoplatonists and gnostics and from the narrow fanaticism which thinks the materialism of Haeckel and Engels the last word of human thought. Will this road lead humanity to the solution of the riddle of reality? Or of that other riddle why and wherefore an insignificant speck of cosmic dust gave birth to human thought? In the history of human thought the page devoted to the problems of space and motion is one of the most important and the most fruitful.

For the solution of this problem let us take for our motto

'LABOREMUS.'

INDEX OF NAMES

A BRAHAM, 159, 160, 161, 183
 Adam, 5
 Albert of Saxony, 27
 Albiruni, 4
 Anaximenes, 2
 Apollonius, 53
 Aquinas, Thomas, 27
 Arago, 137
 Archimedes, 10, 11, 40, 53
 Archytas, 3, 4, 5, 11
 Aristarchus of Samos, 212
 Aristotle, 4, 7, 12, 14, 15, 16,
 18-26, 29-32, 38-42, 45, 49,
 53, 77, 78, 84, 93, 146, 200,
 201, 211, 212
 Aston, 163, 164
 Augustine, 16
 Avenarius, 84, 109, 225
 Averroes, 26, 27, 29, 84, 201

B ACHAM, 193
 Bain, 122
 Bakounin, 81
 Baliana, 48
 Balmes, 213
 Beltrami, 101
 Benedetti, 44, 45, 46, 49, 183
 Bentley, 182
 Bergson, 148
 Berkeley, 64-68, 72, 73, 79, 83,
 85, 88, 89, 112, 113, 114, 121,
 125, 128, 179, 202, 203, 205,
 206, 210, 213, 226
 Bernouilli, John, 53
 Bessel, 95, 183
 Bitault, 14
 Boccaccio, 27
 Bohr, Niels, 160
 Boltzmann, 132
 Bolyai, J., 95
 Bolzano, 51
 Boscovitch, 71
 Bouasse, 214
 Brown, 122
 Bruno, Giordano, 33
 Bucherer, 160
 Buisson, 193

C AMPANELLA, 33, 201
 Campanus of Novara, 28
 Cantor, Georg, 8, 51
 Cardan, 27, 32, 44, 45, 49
 Carnot, 112
 Cassirer, 29, 75, 218, 219, 227
 Cauchy, 82
 Cavalieri, 11
 Cayley, 123, 124
 Chernoushevski, 81
 Christoffel, 125
 Cicero, 12
 Clifford, W. K., 90, 109, 110,
 111, 204, 207
 Comte, Auguste, 85
 Couturat, 125
 Copernicus, 30, 47, 145, 201,
 217
 Cremonini, 29
 Crommelin, 192
 Cudworth, 35

D AMASIUS, 24, 26, 27, 84
 Dante, 26
 De Claves, 14
 Delboeuf, 214
 Democritus, 2, 9, 10, 11-18, 21,
 30, 78, 199-201, 225
 Desargues, 122
 Descartes, 10, 19, 34, 35, 49,
 50, 51, 73, 78, 95, 96, 203,
 224, 227
 De Sitter, 137, 195
 Diogenes Laertius, 13
 Dionysius of Alexandria, 12
 Döppler, 157
 Drude, 182
 Dubois-Reymond, 63
 Duhem, 5, 16, 19, 25, 30, 40,
 41, 42, 44, 90
 Dumas, 75
 Duns Scotus, 27
 Dupuis, 5

E DDINGTON, 150, 192, 219,
 220, 221, 224
 Edwards, Jonathan, 81

Eichenwald, 135, 136
 Einstein, 62, 91, 96, 141, 142,
 145-148, 154, 157, 161, 162,
 163, 165, 173, 181, 183, 184,
 188-196, 206, 208, 218, 221,
 222, 223, 225, 226, 227
 Eleatics, 6, 17, 32
 Engel, 94, 115
 Engels, 6, 228
 Eötvös, 183
 Epicurus, 12, 14, 15
 Erdmann, Benno, 125
 Euclid, 26, 76, 92, 93, 94, 101, 108,
 114, 118, 120, 128, 169, 210
 Eudemus, 4
 Eudoxus of Cnidus, 11
 Euler, 53, 54, 69-72, 74, 88,
 96, 133, 202, 206
 Eusebius, 12
 Evershed, 193

FABRY, 193
 Faraday, 63, 132, 207
 Fechner, 85
 Feuerbach, 6
 Fichte, 82, 204
 FitzGerald, 137, 140, 141, 160,
 214
 Fizeau, 134
 Foucault, 87
 Fraser, 64
 Frederick II, 27
 Fresnel, 133, 134, 157

GALILEO, 10, 20, 29, 30, 34,
 39, 41, 42, 44, 46-51, 55, 57,
 58, 61, 83, 177, 201
 Galois, 58
 Gassendi, 15, 34, 78
 Gauss, 82, 95, 96, 97, 99, 100,
 103, 104, 105, 204, 205
 Gemistos Pletho, 29
 Glitscher, 161
 Goethe, 8, 226
 Grassmann, 104, 205
 Grebe, 193

HAECKEL, 228
 Halsted, 111

Hamilton, 189
 Harzer, 195
 Hegel, 6, 82
 Helmholtz, 80, 82, 83, 85, 114,
 115, 116, 121, 122, 125, 126,
 127, 131, 136, 205, 206
 Heraclitus, 2
 Herbart, 85, 104
 Hering, 122
 Hermann, 69
 Herodotus, 211
 Hertz, 133, 135, 136, 137
 Hertzen, 81
 Hilbert, 101, 189, 217, 223
 Hipparchus, 102
 Hobbes, 27, 78
 Hull, 133
 Hume, 72, 85, 203
 Hupka, 160
 Huygens, 133

ILINE, 81
 Ibn Rushd (Averroes), 26, 27,
 29, 84, 201

JEAN DE JANDUN, 28, 29
 Jenssen, 15
 Jerome, 14
 John Philopon, 30, 31, 41
 Johnson, Samuel, 81
 Jordanus Nemorarius, 42

KANT, 10, 72-85, 101, 103,
 114, 115, 126, 127, 179,
 202, 203, 204, 206, 213,
 227
 Kelvin, Lord, 82, 135
 Kepler, 30, 34, 190, 191
 Kirchhoff, 90
 Klein, F. 115, 123, 124, 217

LAGRANGE, 44, 46, 54
 Lambert, 95
 Langevin, 155
 Laplace, 63, 73, 188, 213
 Lavrov, 81
 Lebedev, 133
 Lechalas, 214
 Legendre, 95

Leibniz, 35, 53, 69, 71-74, 79,
119, 203, 227
Lenard, 217
Lenin, 81
Leonardo da Vinci, 10, 30, 39,
42-45, 49, 122
Le Roy, 146
Leverrier, 191
Levi-Civita, 107, 125, 217
Lie, Sophus, 60, 115, 116, 119,
120, 121, 125, 205
Lipschitz, 125
Littrov, 149
Lobachevski, 39, 82, 93, 95,
100-104, 109-112, 120, 125,
127, 165, 169, 187, 204, 207
Lodge, 217
Lorentz, 124, 136, 138-141, 143,
144, 145, 147, 154, 158, 160,
161, 165, 183, 208, 214, 217
Loria, 5
Lucretius, 12, 13, 14
Lyon, 81

MACH, ERNST, 41, 84-91,
109, 125, 126, 127, 180,
181, 206, 207, 213, 225, 226
Maclaurin, 53, 69
Magnenus, 15
Marx, 6, 12
Maupertuis, 75
Masaryk, 81
Maxwell, 63, 82, 132, 135, 139,
159, 207, 210, 223
Menelaus, 102
Mersenne, 49
Meusnier, 96
Michael Scot, 28
Michelson, 136, 137, 138, 208,
211
Mie, 189, 223
Miller, 137
Minding, 100, 101
Minkowski, 128, 157, 165-169,
173, 183, 209, 212, 215, 221
Möbius, 123
Monge, 96
More, Henry, 31, 35, 36, 37, 201
Morley, 137

Müller, J., 122
Mutakallimun, 14

NAVIER, 82
Neumann, F., 131
Neumann, Karl, 83, 84, 88
Newton, Sir Isaac, 20, 21, 31,
36, 37, 38, 42, 48, 49, 51, 53,
54, 55, 57, 58, 61, 64-68, 69,
71-74, 76-79, 87, 89, 121, 172,
176-184, 187, 188, 190, 191,
194, 201, 206, 213, 225, 227
Nicholson, 133
Nicolas Autricurius, 14
Nicolas Cusanus, 15, 29, 30, 42
Nietzsche, 8
Noble, 138
Nordström, 183

OCCAM, WILLIAM of, 27
Odoevski, 81
Oersted, 131
Oresme, Nicholas, 96
Osiander, A., 217

PARMENIDES, 6, 7, 10
Paschen, 161
Patricio (Patritius), 32, 33, 201
Pearson, 90
Pekár, 183
Pericles, 7
Pérot, 193
Perrin, 87
Petzoldt, 226
Philo of Alexandria, 16
Philolaos, 2, 6
Philopon, 30, 31, 41
Planck, 86, 87, 90, 148, 160
Plato, 1, 3, 5, 6, 9, 10, 11, 14-18,
21, 29, 30, 50, 64, 78, 93, 199,
200, 225
Plekhanov, 81
Pliny, 11
Plotinus, 16
Poincaré, H., 90, 121, 122, 123,
125, 126, 127, 129, 140, 143,
144, 145, 157, 183, 206
Poisson, 82, 188
Poncelet, 122

Proclus, 1, 94
 Protagoras, 8, 9, 64, 78, 198,
 200, 218, 226
 Prout, 163, 164
 Ptolemy, 26, 102, 145, 213
 Pythagoras, Pythagoreans, 1-6,
 8, 9, 10, 16, 18, 92, 93, 98,
 198, 199, 200, 218, 224

RATNOVSKI, 160
 Ravaissou, 42
 Reinaud, 4
 Renan, 1
 Renouvier, 214
 Riemann, 82, 103-110, 116, 118,
 120, 125, 169, 174, 186, 188,
 189, 194, 204, 205, 207, 221
 Ritz, 137
 Roberval, 39
 Roemer, 150
 Roland, 135
 Röntgen, 136
 Russell, 125
 Rutherford, 87, 163

SABASHLIKOV, 43
 Saccheri, 39, 94, 102
 St John, 193
 Scaliger, 32
 Schelling, 82, 204
 Schopenhauer, 75
 Schuppe, 84
 Scot, Michael, 28
 Seeliger, 194
 Sextus Empiricus, 9, 13
 Simonov, 149
 Simplicius, 3, 24, 25, 26, 27, 84
 Socrates, 2, 7, 80
 Soloviev, V. S., 75, 81
 Sommerfeld, 160, 161
 Sosicles, of Corinth, 211
 Spencer, 122
 Spinoza, 203
 Stäckel, 94
 Stallo, 90, 110

Staudt, 122
 Stevinus, Simon, 39, 45
 Stobaeus, 2
 Stokes, 134, 137
 Strutt, 163
 Stumpf, 122
 Swarzschild, 195
 Swedenborg, 75

TANNERY, P., 8, 93
 Tartaglia, 44
 Taurinus, 95
 Telesius, 32, 45
 Tempier, Etienne, 27
 Thales, 2
 Theodosius, 102
 Thomson, Sir J. J., 159
 Thomson, W. (Lord Kelvin)
 Torricelli, 45
 Trouton, 138

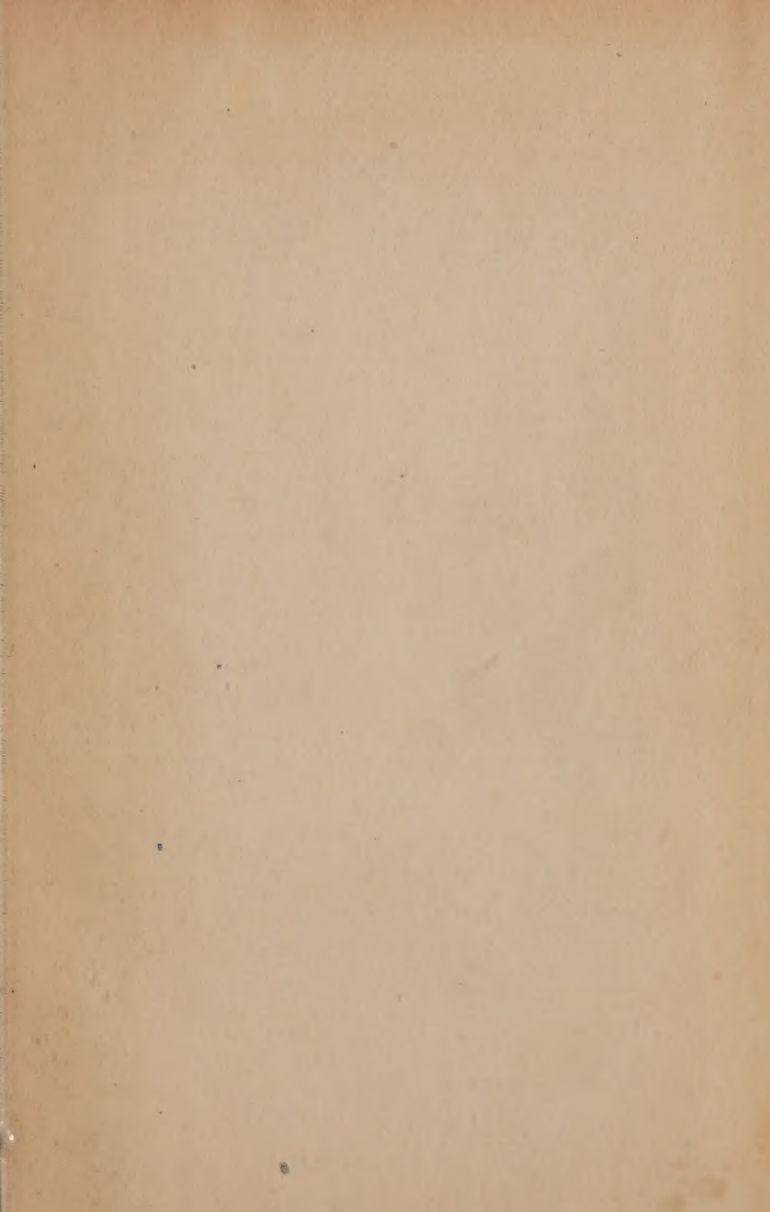
UEBERWEG, 113, 115, 118,
 120, 125, 205

VAILATI, 40, 45
 Varignon, 39
 Vasiliev, N. A., 198
 Venturi, 43
 Villon, 14
 Voight, 139, 208

WALLIS, 94
 Weber, 131
 Weyl, 91, 105, 107, 189, 217,
 221, 222, 223
 Whitehead, 148
 Wilson, 136
 Wohlwill, 41
 Wolff, 71, 72, 74

YOUNG, 133

ZENO, 7, 8, 20, 21, 93, 199,
 200, 212



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